Optimal Central Bank Lending

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Abstract

We analyze optimal monetary policy in a sticky price model with open market operations. The central bank sets the policy rate and can, additionally, control the amount of money by rationing money supplied against eligible securities. Optimal policy under money rationing is shown to enhance welfare in the long-run and in the short-run compared to a conventional optimal policy regime where money supply is not rationed and satiates money demand. Specifically, this property is shown to apply when privately issued debt is eligible in open market operations, which allows the central bank to separately alter costs of borrowing and the size of transactions for which money is required.

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1 Introduction

In the macroeconomic literature of the last two decades, the implementation of monetary policy has focussed on how a risk-free short-term nominal interest rate should be set.[2] Money supply is then passively adjusted by the central bank to satiate money demand, which means that money is supplied until the private agents' marginal valuation of money accords to the marginal costs of holding money in equilibrium. Yet, central banks typically (i.e. in non-crisis times) refrain from fully accommodating money demand, as, for example, the US Federal Reserve "has created what is called a 'structural deficiency'. That is, it has created permanent additions to the supply of reserve balances that are somewhat less than the total need" such that the open market "desk is in a position to add balances temporarily to get to the desired level".[3] Likewise, the European Central Bank has in general not fully accommodated liquidity demand of counterparties by applying "allotment rates" less than one for its main refinancing operations.[4] This indicates that central banks can in fact control both, the nominal interest rate (or, policy rate) as the price of money and the amount of money, by rationing the supply of reserves.[5]

This paper shows that a central bank can enhance welfare by rationing the amount of money supplied in open market operations. Accounting for the fact that reserves are supplied against eligible assets, we consider a collateral constraint for open market operations, where the concept of collateral is used here – like by central banks – in a broader sense and refers to the property of repurchase agreements being a form of collateralized lending. When the central bank supplies money at a price below the marginal valuation of money by private agents, the latter are willing to acquire money against eligible assets until the collateral constraint becomes binding. The policy rate is then decoupled from the marginal rate of intertemporal substitution, which frees up instruments that can change private agents' access to money and interest rates on eligible debt securities. Under money rationing, the central bank can thus control the amount of money as well as interest rates, which allows to separately induce changes in the size of transactions for which money is required and in borrowing costs. Compared to a monetary policy regime that fully accommodates money demand, the central bank can therefore reduce distortionary effects on the allocation more effectively under money rationing simply by having additional instruments at its disposal. This is demonstrated by applying a stylized macroeconomic model, where money rationing is shown to enhance welfare in the long-run and in the short-run.

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[2] See, e.g, several chapters in [12]. Exceptions are analyses of unconventional monetary policies that are applied in crisis times, like [10], [13], or [14].

[3] See Fedpoint "Open Market Operations" at http://www.newyorkfed.org/aboutthefed/fedpoint/fed32.html.

[4] Details on the European Central Bank's allotment rate decisions can be found in [11].

[5] Responses of US Federal Reserve and the European Central Bank to the recent financial crisis, i.e. setting interest rates close to zero and actively expanding the supply of reserves via lending facilities and direct asset purchases, also indicate that central banks can simultaneously control interest rates and the quantity of money.

We examine optimal monetary policy in a framework with frictions that are standard in the literature (see [17], [20], [6], or [23]). Specifically, we allow for goods prices to be set in an imperfectly flexible way, for transaction frictions (i.e. cash constraints), and for time varying mark-ups. In this framework, we explicitly consider that money is supplied by the central bank only in exchange for eligible assets. Since the latter serve as (imperfect) substitutes for money, the interest rates on eligible debt securities relate to the price of money in open market operations and thus to the policy rate. For assets that are non-eligible, investors then demand interest rates that are higher due to an illiquidity premium.[6] These interest rates relate – as usual – to the nominal marginal rate of intertemporal substitution, which reflects the opportunity costs of money holdings and, therefore, determines the private agents' marginal valuation of money. When the policy rate is set at a lower level, private agents are willing to hold money up to the maximum amount supplied by the central bank against eligible assets, such that money supply is effectively rationed.

Money rationing becomes a relevant option for a welfare maximizing central bank if it cannot implement the first best allocation under a conventional single instrument regime. If, for example, only transaction frictions are present, satiating money demand at zero nominal interest rates implements first best, such that money rationing would be undesirable in this case. If, however, efficiency requires a positive nominal marginal rate of intertemporal substitution, e.g. when nominal rigidities call for price stability, rationing money supply is advantageous: Suppose that the central bank sets – like in a conventional regime – the policy rate equal to the nominal marginal rate of intertemporal substitution, such that money supply is not rationed and the central bank has only one instrument at its disposal. Unless transaction frictions are not present and the special case of "divine coincidence" applies,[7] first best will then not be implementable. Now suppose that the central bank sets the policy rate below the desired nominal marginal rate of intertemporal substitution and constrains access to money by a set of eligible assets that are not abundantly available.[8] Then, money supply is rationed, while private agents are not worse off than under a conventional regime. The central bank can in this case deal with multiple distortions, i.e. it can simultaneously stabilize prices and vary the costs of borrowing (to reduce supply side distortions), by separately adjusting the supply of money to manipulate aggregate demand and altering the interest rates on eligible debt securities. Hence, there are gains from money rationing when there exist frictions that, on the one hand, call for non-zero nominal interest rates and, on the other hand, cannot be neutralized just by steering aggregate demand with a single instrument.

[6] This is consistent with empirical evidence by [18] on the yield spread between corporate bonds and treasuries.

[7] The term divince coincidence has been used by [3] to describe the property of a standard New Keynesian model which does not feature a conflict between the goals of an optimizing central bank.

[8] If money were instead supplied in an unconstrained way (e.g. against abundantly available collateral), the nominal marginal rate of intertemporal substitution would then fall to the level of the policy rate, implying inefficient levels of inflation and real activity.

For the analysis of optimal monetary policy, we apply a model that is sufficiently simple to facilitate the derivation of closed form results, while it accounts for some main characteristics of central bank practice. Specifically, we consider money being supplied outright against treasury securities and under repurchase agreements (repos), which lead to private sector money holdings as well as positive money injections. In addition to treasury securities, we also consider corporate loans, which are issued by firms who rely on working capital, as eligible assets for money supply operations.[9] Under money rationing, the central bank then alters the distortive loan rate via policy rate adjustment, while it can separately control the amount of money supplied against collateral in open market operations. Given that supply side distortions exist (implying that there is no divine coincidence), money rationing enables the central bank to enhance welfare by influencing firms' borrowing costs and by simultaneously implementing a desired level of aggregate demand with distinct instruments.[10]

We compare optimal policy under commitment, which is shown to be associated with money rationing, with a conventional optimal policy regime where money supply is not rationed. The latter regime is characterized by the identity of the policy rate and the nominal marginal rate of intertemporal substitution. Decoupling these rates under money rationing endows the central bank with more than one instrument and allows to control the long-run inflation rate independently from the policy rate. The central bank can therefore reduce welfare costs of price stickiness by stabilizing the price level via money supply adjustments and it can simultaneously set the policy rate to reduce the costs of borrowing money; this strategy being impossible under a conventional regime where the policy rate is linked to inflation in the long-run by the Fisher equation. Under a non-rationed money supply, borrowing costs therefore tend to be higher and long-run welfare losses (compared to first best) are a multiple of the long-run welfare losses under an optimal policy regime with money rationing. We further show that effects of cost-push shocks, which are typically considered in the New Keynesian literature and found to substantially contribute to business cycle fluctuations (see [24]), can be neutralized under money rationing. More precisely, the central bank can neutralize effects on firms' marginal costs by lowering the policy rate or increasing the fraction of eligible loans, and can simultaneously adjust the total amount of accepted collateral to induce a level of aggregate demand that closes the output gap. For a special case, where the distortion between cash goods and credit goods is eliminated, it can be shown that money rationing even

[9] The US Federal Reserve mainly accepted "Treasuries only" in pre-2008 open market operations. However, corporate debt securities – like commercial papers that relate to intraperiod loans in the model – have also been considered as substitutes for treasury debt in case of "large budget surpluses and the associated steep reductions in Treasury debt" (see [4]).

[10] We further examine an alternative model version, where we account for heterogenous households who borrow/lend for consumption purposes and we consider consumption loans as eligible assets (see Appendix F). For this alternative version, we also find that monetary policy can enhance welfare via money rationing, though, the welfare gains are less pronounced than for the benchmark model.

enables implementing the first best allocation.[11]

The paper relates to studies on optimal monetary policy under commitment, in particular, with sticky prices and transaction frictions, like [17], [20], and [6]. These studies show that the central bank should predominantly stabilize prices and deviate from the Friedman rule (see [23], for an overview). Optimal policy is also mainly characterized by price stability when prices are sticky and taxes are distortionary, even though inflation serves as a substitute for taxation (see [21] and [2]). For the case where sufficiently many tax instruments are available, [9] show how the effects of price stickiness can be neutralized, while [1] show that the central bank can off-set effects of transaction frictions and of pre-set prices when it simultaneously controls the policy rate and the money growth rate. The paper further relates to studies where a central bank is assumed to directly lend to the private sector, rather than to use private debt as collateral (as assumed in this paper). Analyzing monetary policy in a model with sticky prices and imperfections in private financial intermediation, [10] show that direct central bank lending is associated with costs that differ from private costs of intermediation and can be beneficial at times of unusual financial distress. Similarly, [13] develop a model where private intermediaries face balance sheets constraints, while the central bank can inelastically raise funds at fixed costs per unit lent to the private sector. Like [14], who augment [13] by introducing idiosyncratic investment risks, they show that direct lending is beneficial in crises situations when private intermediaries are financially constrained. In contrast to these studies, we do not consider credit origination by the central bank and abstract from costs of central bank operations.

The paper is organized as follows. Section 2 presents the benchmark model. In Section 3, we describe the role of money rationing. In Section 4, we describe the policy problem under commitment and present some closed form results for a special case. Section 5 provides comparisons to a conventional optimal policy regime without money rationing. In Section 6, we discuss limits to money rationing and differences to direct central bank lending. Section 7 concludes.

2 The model

In this Section, we present the model, which features frictions that are standard in the New Keynesian literature on monetary policy, namely, sticky prices, potentially time varying mark-ups, and transaction frictions. The latter are modelled by liquidity constraints for households, who rely on money for purchases of a cash good (in contrast to purchases of a credit good), and firms, who rely on working capital. The main difference to standard models is that money is supplied only in exchange for assets in open market operations. There, the central bank controls the price of money

[11] Notably, optimal monetary policy under money rationing is not limited to policy rate adjustments, which is most apparent at the zero lower bound where private sector behavior can still be affected via money supply instruments, as shown by [16] in a companion paper.

and the amount of money supplied against eligible assets, which consist of short-term treasuries and corporate loans. By deciding on how much money is supplied against these assets, the central bank influences their prices compared to non-eligible assets, since agents internalize the property of assets to serve as (imperfect) substitutes for money. Following common central bank practice, we assume that money is supplied outright as well as under repurchase agreements (repos), which ensures positive money injections in each period and that central bank transfers consist of total interest earnings.

2.1 Timing of events

Households, indexed with $i \in [0,1]$, enter a period t with money $M^H_{i,t-1}$, one-period government bonds $B_{i,t-1}$, and contingent claims $D_{i,t}$. After aggregate shocks are realized at the beginning of the period, the central bank sets its instruments, i.e. it announces the maximum amount of money as a fraction of eligible assets held by the counterparty (κ^B_t and κ_t, see below), and sets the policy rate R^m_t. The remainder of the period unfolds as follows.

First, the labor market opens, where a perfectly competitive intermediate goods producing firm j hires workers $n_{j,t}$. We assume that it has to pay wages before the goods are sold. Since it does not hold any financial wealth, firm j borrows working capital to finance its wage bill

$$L_{j,t}/R^L_{j,t} \geq P_t w_t n_{j,t}, \tag{1}$$

where w_t denotes the aggregate real wage rate and P_t the final goods price. Firm j borrows the amount $L_{j,t}/R^L_{j,t}$ and repays the amount $L_{j,t}$ at the end of the period, such that $R^L_{j,t}$ is the interest rate on the intraperiod loan.

Second, open market operations are conducted, where the central bank supplies money outright or under repos against eligible assets at the price R^m_t. Household i receives new money (injections) from the central bank $I_{i,t}$ against eligible assets, i.e. corporate loan contracts and government bonds. Specifically, the central bank supplies money against fractions of randomly selected treasuries κ^B_t and loan contracts κ_t, such that $I_{i,t}$ is constrained by the following condition, which we summarize as the 'collateral constraint':[12]

$$I_{i,t} \leq \kappa^B_t \left(B_{i,t-1}/R^m_t\right) + \kappa_t \left(L_{i,t}/R^m_t\right). \tag{2}$$

After receiving $I_{i,t}$, household i delivers $L_{i,t}/R^L_t$ to firms according to the loan contracts. It then holds money, bonds, and loans to the amount $M^H_{i,t-1} + I_{i,t} - (L_{i,t}/R^L_t)$, $B_{i,t-1} - \Delta B^c_{i,t}$, and $L_{i,t} - L^R_{i,t}$, where $\Delta B^c_{i,t}$ are treasuries received by the central bank and $L^R_{i,t}$ are loans under repos, such that

[12] Though, the term collateral only applies to repos and not to outright purchases, it is – like by central banks – used in a broader sense throughout the paper, for convenience.

$I_{i,t} = (\Delta B^c_{i,t}/R^m_t) + (L^R_{i,t}/R^m_t)$.

Third, wages are paid, and intermediate as well as final goods are produced. Then, the goods market opens, where cash goods c_t – in contrast to credit goods $\widetilde{c}_t$ – can only be purchased with money. Hence, household i faces the 'cash-in-advance constraint'

$$P_t c_{i,t} \leq I_{i,t} + M^H_{i,t-1} - \left(L_{i,t}/R^L_t\right) + P_t w_{i,t} n_{i,t}, \tag{3}$$

where $w_{i,t}$ denotes the individual wage rate. Household $i's$ stock of money then equals $\widetilde{M}_{i,t} = M^H_{i,t-1} + I_{i,t} - (L_{i,t}/R^L_t) + P_t w_{i,t} n_{i,t} - P_t c_{i,t} \geq 0$ and its stock of treasuries equals $\widetilde{B}_{i,t} = B_{i,t-1} - \Delta B^c_{i,t} \geq 0$.

Fourth, before household i enters the asset market, it receives government transfers $P_t \tau_{i,t}$, and dividends of firms and retailers, which sum up to $P_t \delta_{i,t}$. Repurchase agreements are then settled, i.e. household i buys back loans $L^R_{i,t} = M^L_{i,t} R^m_t$ and treasuries $B^R_{i,t} = M^R_{i,t} R^m_t$ from the central bank, where $M^L_{i,t}$ and $M^R_{i,t}$ denote money supplied temporarily against loans and treasuries. In the asset market, returns from maturing assets are paid, loans are repaid, and treasuries are issued (at the price $1/R_t$) as well as contingent claims. Household $i's$ asset market constraint is thus given by

$$\begin{aligned} &(B_{i,t}/R_t) + E_t[q_{t,t+1} D_{i,t+1}] + M^H_{i,t} \\ &\leq \widetilde{B}_{i,t} + B^R_{i,t} + \widetilde{M}_{i,t} - (M^L_{i,t} + M^R_{i,t}) R^m_t + L_{i,t} + D_{i,t} + P_t \widetilde{c}_{i,t} + P_t \tau_{i,t} + P_t \delta_{i,t}, \end{aligned} \tag{4}$$

where $q_{t,t+1}$ denotes a stochastic discount factor (see Section 2.3). The central bank transfers seigniorage to the treasury and reinvests payoffs from maturing bonds in newly issued bonds and leaves aggregate money supply unchanged at this stage, $\int_0^1 M^H_{i,t} di = \int_0^1 (M^H_{i,t-1} + I_{i,t} - M^L_{i,t} - M^R_{i,t}) di$.

2.2 Firms

There is a continuum of identical intermediate goods producing firms indexed with $j \in [0,1]$. They exist for one period, are perfectly competitive, and are owned by the households. A firm j distributes profits to the owners and hires the aggregate labor input $n_{j,t}$ at a common wage rate w_t. We assume that wages have to be paid before goods are sold. For this, firm j borrows cash $L_{j,t}$ from households at the price $1/R^L_{j,t}$ (see 1) and repays the loan at the end of the period.[13] It then produces the intermediate good according to $IO_{j,t} = a_t n^{\alpha}_{j,t}$, where $\alpha \in (0,1)$ and a_t is stochastic with an unconditional mean equal to one, and sells it to retailers. Following related studies (see e.g. [6] and [23]), we consider a constant subsidy τ^p to eliminate long-run distortions, such that

[13] In Appendix D, we show that the possibility of retained earnings does not to affect our main results.

the problem of a profit-maximizing firm j is given by

$$\max_{\{n_{j,t},l_{j,t}\}}(1+\tau^{p})P_{J,t}a_{t}n_{j,t}^{\alpha}-P_{t}w_{t}n_{j,t}-L_{j,t}(R_{j,t}^{L}-1)/R_{j,t}^{L},\quad \text{s.t. (1)}, \tag{5}$$

where $P_{J,t}$ denotes the price for the intermediate good. The first order conditions are given by $(1+\tau^{p})\left(P_{J,t}/P_{t}\right)\alpha n_{j,t}^{1-\alpha} = w_{t}+\varsigma_{j,t}w_{t}$, $R_{t}^{L}-1=\varsigma_{j,t}$, and $\varsigma_{j,t}[(l_{j,t}/R_{t}^{L})-w_{t}n_{j,t}]=0$, where $l_{j,t}=L_{j,t}/P_{t}$ and $\varsigma_{j,t}\geq 0$ is the multiplier on (1). Defining $\tau^{n}=\tau^{p}/(1+\tau^{p})$ as the production (or wage) subsidy rate, labor demand and loans satisfy:

$$\left(P_{J,t}/P_{t}\right)a_{t}\alpha n_{j,t}^{\alpha-1}=\left(1-\tau^{n}\right)w_{t}R_{j,t}^{L}, \tag{6}$$

$$l_{j,t}/R_{j,t}^{L}=w_{t}n_{j,t}, \tag{7}$$

if $R_{t}^{L}>1$, while $(l_{j,t}/R_{t}^{L})\geq w_{t}n_{j,t}$ holds instead of (7) if $R_{t}^{L}=1$. Firms transfer profits to the owners in a lump-sum way. Condition (6) shows that the working capital constraint (1) can distort labor demand through the costs of borrowing R_{t}^{L}. Since intermediate goods producing firms are ex-ante identical, they can only exhibit different labor demands if they face different costs of borrowing (6). This would, for example, be the case, if lenders perceive loans of different firms as imperfect substitutes. We exclude this possibility by assuming that the central bank treats loans of all firms in an ex-ante identical way (see Section 6 for a discussion).

To introduce sticky prices, we assume that there are monopolistically competitive retailers who re-package intermediate goods $IO_{t}=\int_{0}^{1}IO_{j,t}dj$. A retailer $k\in[0,1]$ produces one unit of a distinct good $y_{k,t}$ with one unit of the intermediate good (purchased at the common price $P_{J,t}$) and sells it at the price $P_{k,t}$ to perfectly competitive bundlers. They bundle the distinct goods $y_{k,t}$ to a final good y_{t} that can be used for consumption as a cash good or as a credit good. Both goods are thus produced according to the same technology $y_{t}=(\int_{0}^{1}y_{k,t}^{\frac{\varepsilon-1}{\varepsilon}}dk)^{\frac{\varepsilon}{\varepsilon-1}}$ and are sold at the same price P_{t}. The cost minimizing demand for $y_{k,t}$ is then given by $y_{k,t}=\left(P_{k,t}/P_{t}\right)^{-\varepsilon}y_{t}$. Following [5], we assume that each period a measure $1-\phi$ of randomly selected retailers may reset their prices independently of the time elapsed since the last price setting, while a fraction $\phi\in(0,1)$ of retailers do not adjust their prices. A fraction $1-\phi$ of retailers sets their price to maximize the expected sum of discounted future profits. For $\phi>0$, the first order condition for their price $\widetilde{P}_{t}$ is given by

$$\widetilde{P}_{t}=\frac{\varepsilon}{\varepsilon-1}\frac{E_{t}\sum_{s=0}^{\infty}\left(\phi\beta\right)^{s}q_{t,t+s}y_{t+s}P_{t+s}^{\varepsilon}mc_{t+s}}{E_{t}\sum_{s=0}^{\infty}\left(\phi\beta\right)^{s}q_{t,t+s}y_{t+s}P_{t+s}^{\varepsilon-1}}, \tag{8}$$

where $mc_{t}=P_{J,t}/P_{t}$ denotes retailers' real marginal cost. With perfectly competitive bundlers, the price index P_{t} for the final good satisfies $P_{t}^{1-\varepsilon}=\int_{0}^{1}P_{k,t}^{1-\varepsilon}dk$. Using that $\int_{0}^{1}P_{k,t}^{1-\varepsilon}dk=(1-\phi)\sum_{s=0}^{\infty}\phi^{s}\widetilde{P}_{t-s}^{1-\varepsilon}$ holds, and taking differences, leads to $P_{t}^{1-\varepsilon}=(1-\phi)\widetilde{P}_{t}^{1-\varepsilon}+\phi P_{t-1}^{1-\varepsilon}$.

2.3 Households

There is a continuum of infinitely lived households indexed with $i \in [0,1]$ and with identical asset endowments and preferences. Instantaneous utility decreases with working time and increases with consumption, which consists of cash goods $c_{i,t}$ and credit goods $\widetilde{c}_{i,t}$. Following [19], we assume that cash and credit goods are distinct from the households' perspective.[14] Household i maximizes the expected sum of a discounted stream of instantaneous utilities

$$E\sum_{t=0}^{\infty}\beta^t u(c_{i,t},\widetilde{c}_{i,t},n_{i,t}), \tag{9}$$

where E is the expectation operator conditional on the information set in the initial period, $\beta \in (0,1)$ is the subjective discount factor, and the utility function is given by $u(c_{i,t},\widetilde{c}_{i,t},n_{i,t}) = [(c_{i,t}^{1-\sigma} - 1) + \gamma(\widetilde{c}_{i,t}^{1-\sigma} - 1)]\,(1-\sigma)^{-1} - \chi n_{i,t}^{1+\eta}\,(1+\eta)^{-1}$, where $\sigma > 0$, $\chi > 0$, $\eta \geq 0$, and $\gamma \geq 0$.

Households are assumed to monopolistically supply differentiated labor services $n_{i,t}$ that are transformed into aggregate labor input n_t employed for the production of intermediate goods. The transformation is conducted via the aggregator $n_t^{1-1/\zeta_t} = \int_0^1 n_{i,t}^{1-1/\zeta_t}di$, where we follow [24] and assume that the elasticity of substitution ζ_t varies exogenously over time according to a stationary process. Cost minimization leads to the following demand for differentiated labor services $n_{i,t}$,

$$n_{i,t} = (w_{i,t}/w_t)^{-\zeta_t}\,n_t, \quad \text{with} \quad w_t^{1-\zeta_t} = \int_0^1 w_{i,t}^{1-\zeta_t}di. \tag{10}$$

A household i is initially endowed with money $M_{i,-1}^H > 0$, government bonds $B_{i,-1} > 0$, and contingent claims $D_{i,0}$. Before household i enters the goods market, where it relies on cash for goods purchases (see 3), it might lend money to firms. It can then acquire money in open market operations up to fractions κ_t and κ_t^B of randomly selected loans and treasuries (see 2). In the goods market, household i can use wages, money holdings net of loans, and additional cash from current period open market operations for its consumption expenditures (see 3). In the asset market, household i receives payoffs from maturing assets, buys treasuries, and borrows/lends using a full set of nominally state contingent claims. Dividing the period t price of one unit of nominal wealth in a particular state of period $t+1$ by the period t probability of that state gives the stochastic discount factor $q_{t,t+1}$. The period t price of a random payoff $D_{i,t}$ from investments in state contingent claims in period $t+1$ is then given by $E_t[q_{t,t+1}D_{i,t+1}]$. Substituting out the stocks of bonds and money held before the asset market opens, $\widetilde{B}_{i,t}$ and $\widetilde{M}_{i,t}$, in (4), the asset

[14] Credit goods are sold on trade-credit, which is neither explicitly specified in this paper nor considered as a pledgeable asset. In Appendix F, we analyze consumption loans in an alternative model version and allow for the possibility that they are eligible in open market operations.

market constraint of household i can be written as

$$
\begin{aligned}
& M^H_{i,t-1} + B_{i,t-1} + L_{i,t}\left(1 - 1/R^L_t\right) + P_t w_{i,t} n_{i,t} + D_{i,t} + P_t \delta_{i,t} + P_t \tau_{i,t} \\
& \geq M^H_{i,t} + (B_{i,t}/R_t) + E_t[q_{t,t+1} D_{i,t+1}] + I_{i,t}\left(R^m_t - 1\right) + P_t c_{i,t} + P_t \widetilde{c}_{i,t},
\end{aligned} \tag{11}
$$

while its borrowing is restricted by the no-Ponzi game condition $\lim_{s\to\infty} E_t q_{t,t+s} D_{i,t+s+1} \geq 0$ as well as by $M^H_{i,t} \geq 0$ and $B_{i,t} \geq 0$. The term $I_{i,t}\left(R^m_t - 1\right)$ in (11) measures the costs of money acquired in open market operations. Maximizing (9) subject to the collateral constraint (2), the cash-in-advance constraint (3), labor demand (10), the asset market constraint (11), and the borrowing constraints, for given initial values $M^H_{i,-1}$, $B_{i,-1}$, and $D_{i,0}$, leads to the following first order conditions for both types of consumption goods, working time, injections, and loans

$$c^{-\sigma}_{i,t} = \lambda_{i,t} + \psi_{i,t}, \tag{12}$$

$$\gamma \widetilde{c}^{-\sigma}_{i,t} = \lambda_{i,t}, \tag{13}$$

$$\mu_t \chi n^{\eta}_{i,t} = w_t\left(\lambda_{i,t} + \psi_{i,t}\right), \tag{14}$$

$$\psi_{i,t} = \left(R^m_t - 1\right)\lambda_{i,t} + R^m_t \eta_{i,t}, \tag{15}$$

$$R^m_t\left(\lambda_{i,t} + \eta_{i,t}\right) = R^L_t\left(\lambda_{i,t} + \eta_{i,t}\kappa_t\right), \tag{16}$$

where $\mu_t = \zeta_t/(\zeta_t - 1)$ denotes a stochastic wage mark-up, $\lambda_{i,t} \geq 0$ the multiplier on the asset market constraint (11), $\eta_{i,t} \geq 0$ the multiplier on the collateral constraint (2), and $\psi_{i,t} \geq 0$ the multiplier on the cash-in-advance constraint (3). Notably, the latter affects the consumption and leisure choices (12) and (14), implying $c^{-\sigma}_{i,t} = \mu_t \chi n^{\eta}_{i,t}/w_t$. Further, the following first order conditions for holdings of bonds, money, and contingent claims, and complementary slackness conditions,

$$\lambda_{i,t} = \beta R_t E_t \frac{\lambda_{i,t+1} + \kappa^B_{t+1}\eta_{i,t+1}}{\pi_{t+1}}, \tag{17}$$

$$\lambda_{i,t} = \beta E_t \frac{\lambda_{i,t+1} + \psi_{i,t+1}}{\pi_{t+1}}, \tag{18}$$

$$q_{t,t+1} = \frac{\beta}{\pi_{t+1}} \frac{\lambda_{i,t+1}}{\lambda_{i,t}}, \tag{19}$$

$$0 = \psi_{i,t}[I_{i,t} + M^H_{i,t-1} - \left(L_{i,t}/R^L_t\right) + P_t w_t n_{i,t} - P_t c_{i,t}], \tag{20}$$

$$0 = \eta_{i,t}[\kappa^B_t B_{i,t-1} + \kappa_t L_{i,t} - R^m_t I_{i,t}], \tag{21}$$

as well as (2), (3), and (11) with equality (since $\lambda_{i,t} > 0$, see 13) and the transversality conditions hold. Combining (15), (17), and (18) to $R_t E_t[\left(\lambda_{i,t+1} + \kappa^B_{t+1}\eta_{i,t+1}\right)/\pi_{t+1}] = E_t[R^m_{t+1}(\lambda_{i,t+1} + \eta_{i,t+1})/\pi_{t+1}]$, shows that household i is indifferent between investing in money or investing in treasuries and converting a fraction κ^B_{t+1} of them into cash in the next period at the rate R^m_{t+1}.

Likewise, (16) shows that the loan rate R_t^L depends on the fraction of loans eligible as collateral in open market operations, κ_t. Using (12), (15), and (18), condition (16) can be written as

$$1/R_t^L = (1-\kappa_t)\, c_{i,t}^{\sigma}\beta E_t[c_{i,t+1}^{-\sigma}\pi_{t+1}^{-1}] + \kappa_t/R_t^m, \tag{22}$$

which shows that the inverse of the loan rate is a convex combination of the inverse of the nominal marginal rate of intertemporal substitution of consumption of the cash good $c_{i,t}^{\sigma}\beta E_t[c_{i,t+1}^{-\sigma}\pi_{t+1}^{-1}]$ and the inverse of the policy rate $1/R_t^m$. When loans are not fully eligible, $\kappa_t < 1$, there can be a spread between the policy rate and the loan rate, i.e. an illiquidity premium, while for $\kappa_t = 1$ both rates are identical, $R_t^L = R_t^m$. Otherwise, $\kappa_t = 0$, the loan rate equals $c_{i,t}^{-\sigma}/(\beta E_t[c_{i,t+1}^{-\sigma}\pi_{t+1}^{-1}])$.

2.4 Public sector

The central bank transfers seigniorage revenues $P_t\tau_t^m$ to the treasury, which issues one-period bonds and pays a subsidy at a constant rate. The supply of short-term government bonds is specified in a simple way. Specifically, we assume that the total amount of short-term treasuries B_t^T, which are either held by households or the central bank, grows at some exogenously determined rate $\Gamma > \beta$,

$$B_t^T = \Gamma B_{t-1}^T, \tag{23}$$

given $B_{-1}^T > 0$. Note that we do not aim to measure total public debt by the stock of short-term bonds B_t^T, which can be interpreted as t-bills. To abstract from further effects of fiscal policy, we assume that the government has access to lump-sum transfers $P_t\tau_t$, which balance the budget. Its budget constraint is thus given by $(B_t^T/R_t) + P_t\tau_t^m = B_{t-1}^T + P_t\tau_t + P_t\tau^p mc_t a_t \int_0^1 n_{j,t}^{\alpha} dj$.

In open market operations, the central bank supplies money outright and temporarily in repos against treasuries, $M_t^H = \int_0^1 M_{i,t}^H di$ and $M_t^R = \int_0^1 M_{i,t}^R di$, and against corporate loans under repos, $M_t^L = \int_0^1 M_{i,t}^L di$. At the beginning of each period, its stock of treasuries equals B_{t-1}^c and the stock of outstanding money equals M_{t-1}^H. It then receives an amount ΔB_t^c of treasuries and loans L_t^R in exchange for money at the amount $(\Delta B_t^c/R_t^m) + (L_t^R/R_t^m)$. Before the asset market opens, where the central bank rolls over maturing assets, repos are settled. Its budget constraint thus reads

$$(B_t^c/R_t) - B_{t-1}^c + P_t\tau_t^m = R_t^m\left(M_t^H - M_{t-1}^H\right) + (R_t^m - 1)\left(M_t^L + M_t^R\right). \tag{24}$$

Accounting for common central bank practice, we assume that the central bank transfers its earnings from holding assets and from open market operations to the treasury, $P_t\tau_t^m = (1-1/R_t)\,B_t^c + (R_t^m - 1)\left(M_t^H - M_{t-1}^H + M_t^L + M_t^R\right)$. Substituting out transfers in (24) shows that central bank asset holdings evolve according to $B_t^c - B_{t-1}^c = M_t^H - M_{t-1}^H$. Assuming that initial values for central bank's assets and liabilities satisfy $B_{-1}^c = M_{-1}^H$, delivers the central bank's balance sheet

$$B_t^c = M_t^H. \tag{25}$$

The central bank has three main instruments. It sets the policy rate $R_t^m \geq 1$ and can decide how much money to supply as fractions of eligible assets, for which it can adjust the two additional instruments, κ_t and κ_t^B, in a state contingent way. We assume that the central bank sets κ_t^B between zero and one, $\kappa_t^B \in (0,1]$ and κ_t larger or equal to zero, $\kappa_t \geq 0$. To account for the case where (1) is not binding (if $R_t^L = 1$), we restrict κ_t to be smaller than the wages-to-loans ratio $\kappa_t \leq \overline{\kappa}_t$, where $\overline{\kappa}_t = R_t^L w_t \int_0^1 n_{j,t}dj / \int_0^1 l_{j,t}dj$, such that $\kappa_t \in [0, \min\{1, \overline{\kappa}_t\}]$. By satisfying $\kappa_t \leq \overline{\kappa}_t$, the central bank only accepts loans up to the amount that is issued for current production expenditures.[15] If (1) is binding, i.e. when $R_t^L > 1$, $\kappa_t \leq \overline{\kappa}_t$ implies $\kappa_t \leq 1$.

Finally, the central bank can decide whether money is supplied in exchange for treasuries via repos or outright (while loans are only traded under repos). Specifically, it controls the ratio of treasury repos to outright purchases of bonds $\Omega_t > 0 : M_t^R = \Omega_t M_t^H$, where a sufficiently large value for Ω_t ensures that injections are always positive, $I_{i,t} > 0$. The ratio Ω_t can further be adjusted in the long-run to implement the desired long-run inflation target (see Proposition 7).

2.5 Equilibrium

We restrict our attention to symmetric equilibria, where all households and intermediate goods producing firms behave in an identical way. There will be no arbitrage opportunities and markets clear, $n_t = \int_0^1 n_{j,t}dj = \int_0^1 n_{i,t}di$, $y_t = \int_0^1 c_{i,t}di + \int_0^1 \widetilde{c}_{i,t}di = c_t + \widetilde{c}_t$, and $\int_0^1 L_{i,t}di = \int_0^1 L_{j,t}dj = L_t$. Aggregate stocks of assets satisfy $\int_0^1 D_{i,t+1}di = 0$, $\int_0^1 M_{i,t}^H di = M_t^H$, $\int_0^1 M_{i,t}^R di = M_t^R$, $\int_0^1 M_{i,t}^L di = M_t^L$, $\int_0^1 B_{i,t}di = B_t$, $\int_0^1 I_{i,t}di = I_t = M_t^H - M_{t-1}^H + M_t^R + M_t^L$, and $B_t^T = B_t + B_t^c$. Using the latter and (25), imply household bond holdings to satisfy $b_t = b_t^T - m_t^H$, where b_t, b_t^T, and m_t^H denote real values of government liabilities $b_t = B_t/P_t$, $b_t^T = B_t^T/P_t$, and $m_t^H = M_t^H/P_t$.

Since intermediate goods producing firms behave in an identical way, their aggregate output satisfies $IO_t = a_t n_t^\alpha$. Retailers can differ with regard to their prices, which might lead to dispersed retail prices. Market clearing for the intermediate goods market, $IO_t = \int_0^1 y_{k,t}dk$ then implies for aggregate output $a_t n_t^\alpha = \int_0^1 (P_{k,t}/P_t)^{-\varepsilon} y_t dk \Leftrightarrow y_t = a_t n_t^\alpha / s_t$, where s_t is a measure of price dispersion, $s_t = \int_0^1 (P_{k,t}/P_t)^{-\varepsilon} dk$. Following [25] and [22], we restrict our attention to the case where there is no initial price dispersion, $s_{-1} = 1$. Taking different cohorts of price adjusting firms into account, s_t can be written as $s_t = (1-\phi)(Z_t/P_t)^{-\varepsilon} + (1-\phi)\phi(Z_{t-1}/P_t)^{-\varepsilon} + (1-\phi)\phi^2(Z_{t-2}/P_t)^{-\varepsilon} + ...$ and therefore as $s_t = (1-\phi)\sum_{l=0}^{\infty} \phi^l \tilde{Z}_{t-l}^{-\varepsilon} \Pi_{s=1}^l \pi_{t+1-s}^\varepsilon$. Taking differences, leads to $s_t = (1-\phi)\tilde{Z}_t^{-\varepsilon} + \phi s_{t-1}\pi_t^\varepsilon$. Defining $\tilde{Z}_t = \widetilde{P}_t/P_t$ and rewriting the denominator and numerator in a recursive way, (8) can be written as $\tilde{Z}_t = \frac{\varepsilon}{\varepsilon-1} Z_{1,t}/Z_{2,t}$, where – using (13) and (19) – $Z_{1,t} = \gamma \widetilde{c}_t^{-\sigma} y_t mc_t + \phi\beta E_t \pi_{t+1}^\varepsilon Z_{1,t+1}$ and $Z_{2,t} = \gamma \widetilde{c}_t^{-\sigma} y_t + \phi\beta E_t \pi_{t+1}^{\varepsilon-1} Z_{2,t+1}$. Aggregation over retail prices further gives $1 = (1-\phi)\tilde{Z}_t^{1-\varepsilon} + \phi\pi_t^{\varepsilon-1}$. A definition of the competitive equilibrium is

[15] In contrast, loans that are not issued by firms to finance the wage bill, which might be the case when (1) is not binding, are not eligible.

given in Appendix A.

3 The role of money rationing

To see how money rationing can affect the equilibrium allocation, consider first the cash-in-advance constraint (3), which implies $P_t c_t \leq M_t^H + M_t^R + M_t^L$ in equilibrium. It gives rise to non-neutrality of monetary policy if it is binding. To see when this is the case, use the conditions (12) and (18), which imply $c_t^{-\sigma} = \beta E_t \frac{c_{t+1}^{-\sigma}}{\pi_{t+1}} + \psi_t$ and that the multiplier on the cash-in-advance constraint ψ_t satisfies in equilibrium

$$\psi_t = c_t^{-\sigma}[1 - (1/\overline{R}_t)] \geq 0, \tag{26}$$

where $\overline{R}_t$ is defined as the nominal marginal rate of intertemporal substitution of cash good consumption, $1/\overline{R}_t = \beta E_t \frac{c_{t+1}^{-\sigma}/P_{t+1}}{c_t^{-\sigma}/P_t}$. Households are indifferent between $1/\overline{R}_t$ units of the means of payment in period t, which is required for consumption purchases, and one unit in period $t+1$. They are therefore willing to give up $\overline{R}_t - 1$ in order to transform one unit of account into one unit of money today. Thus, a positive value $\overline{R}_t - 1 > 0$ reflects – like in standard models – a strictly positive valuation for money and implies that households will not hold more money than needed for consumption expenditures. Then, $\psi_t > 0$ (see 26) and the cash-in-advance constraint is binding (see 20), while it is not binding, $\psi_t = 0$, if $\overline{R}_t = 1$.

Now consider the collateral constraint (2), which in equilibrium reads

$$M_t^H - M_{t-1}^H + M_t^L + M_t^R \leq \kappa_t^B \left(B_{t-1}/R_t^m\right) + \kappa_t \left(L_t/R_t^m\right). \tag{27}$$

Notably, the instruments κ_t^B and κ_t enter the set of equilibrium conditions (see Definition 1 in Appendix A) only via the collateral constraint (27) and the asset pricing conditions (16) and (17), and jointly with the multiplier η_t. Hence, if (27) is not binding, $\eta_t = 0$, the instruments κ_t^B and κ_t will not affect the equilibrium allocation and prices, and the model reduces to a standard model (see Definition 2 in Appendix A). To see when (27) is binding, use that (12), (15), and (18) imply $c_t^{-\sigma} = R_t^m(\lambda_t + \eta_t)$ and $\lambda_t = \beta E_t \frac{c_{t+1}^{-\sigma}}{\pi_{t+1}}$ in equilibrium, and eliminate λ_t, which leads to the following condition for the multiplier on the collateral constraint η_t

$$\eta_t = c_t^{-\sigma}[(1/R_t^m) - (1/\overline{R}_t)] \geq 0. \tag{28}$$

Condition (28) shows that, when the policy rate R_t^m is smaller than $\overline{R}_t$, the multiplier η_t is strictly positive and the collateral constraint is binding (see 21), such that access to money depends on the instruments $\kappa_t^{(B)}$ and on holdings of eligible assets. In this case, the cash-in-advance constraint is binding as well, $\psi_t > 0$ (see 26), given that $R_t^m \geq 1$. Households can then get money in exchange for eligible assets at the price, $R_t^m - 1$, which is below their marginal valuation of money, $\overline{R}_t - 1$. Hence, they use eligible assets as much as possible to get money in open market operations, such that (27) is binding and money supply is effectively rationed by the central bank's collateral requirements.[16]

To demonstrate how agents' decisions are distorted and how the policy instruments can be used to address these distortions, we will repeatedly refer to the first best allocation, which is characterized in the following Proposition.

Proposition 1 *The first best allocation* $\{c_t^*, \widetilde{c}_t^*, n_t^*\}_{t=0}^{\infty}$ *satisfies*

$$a_t(n_t^*)^\alpha = c_t^* + \widetilde{c}_t^*, \;\; \chi\,(n_t^*)^{1+\eta-\alpha}\,(c_t^*)^\sigma = a_t\alpha, \quad \text{and} \;\; \gamma\,(\widetilde{c}_t^*)^{-\sigma} = (c_t^*)^{-\sigma}\,, \tag{29}$$

and thus $n_t^* = [a_t^{1-\sigma}\left(1+\gamma^{1/\sigma}\right)^\sigma(\alpha/\chi)]^{\frac{1}{\eta+1-\alpha+\sigma\alpha}}$, $c_t^* = a_t\,(n_t^*)^\alpha / \left(1+\gamma^{1/\sigma}\right)$, *and* $\widetilde{c}_t^* = c_t^*\gamma^{1/\sigma}$.

Proof. See Appendix B ■

The model exhibits several frictions, which are standard in the literature on optimal monetary policy (see [6] or [23]). *First*, prices are imperfectly flexible and can be dispersed, $s_t > 1$, which leads to an inefficient allocation of working time. *Second*, imperfect competition between labor suppliers and between retailers lead to mark-ups over wages and over prices, i.e. μ_t and $1/mc_t$, which can vary over time due to innovations to the substitution elasticity ζ_t and due to sticky prices. *Third*, firms rely on loans to finance wages in advance, such that labor demand is distorted by the loan rate if $R_t^L > 1$. *Fourth*, the households' cash-in-advance constraint distorts the choice of consuming cash goods c_t or credit goods $\widetilde{c}_t$ if $\overline{R}_t > 1$. The effects of these distortions can easily be identified by comparing (29) with the following equilibrium conditions that immediately follow from (6), (12)-(14), $y_t = a_t n_t^\alpha / s_t$, and the resource constraint:

$$a_t n_t^\alpha = (c_t + \widetilde{c}_t)\cdot s_t, \tag{30}$$

$$\chi n_t^{1+\eta-\alpha} c_t^\sigma = a_t\alpha\cdot\left[\frac{mc_t}{\mu_t\,(1-\tau^n)}\frac{1}{R_t^L}\right], \tag{31}$$

$$\gamma\widetilde{c}_t^{-\sigma} = \beta E_t\left[c_{t+1}^{-\sigma}/\pi_{t+1}\right], \tag{32}$$

A conventional interest rate policy that is associated with a non-rationed money supply cannot implement the first best allocation, which is a well-established result in the literature on optimal monetary policy in sticky price models (see [8], [17], or [6]). This result holds true even if not all distortions mentioned above were present, since the central bank has only one instrument (R_t^m) at its disposal. If, however, the central bank supplies money subject to the collateral constraint (27) and the latter is binding, additional instruments (i.e. κ_t^B and κ_t) are available.[17] Then, money supply is rationed and the policy rate as well as interest rate on eligible assets (i.e. R_t and

[16] This further implies that R_t^m cannot be larger than $\overline{R}_t$ (see 28).

[17] The instrument Ω_t is used to ensure positive injections, $I_t > 0$, and to implement the long-run inflation target (see Proposition 7), while it will not be utilized for stabilization purposes.

R_t^L) are decoupled from the marginal rate of intertemporal substitution, which reflects changes in consumption and inflation that can be influenced by the central bank's supply of money. Money rationing thus allows freeing up money supply instruments in addition to the policy rate and enables the central bank to address more than one distortion. The following Proposition summarizes how monetary policy instruments affect the determination of the equilibrium allocation under both scenarios.

Proposition 2 *For given sequences* $\{s_t,\ \pi_t\}_{t=0}^{\infty}$, *the equilibrium sequences* $\{c_t, \widetilde{c}_t, n_t, mc_t, R_t^L\}_{t=0}^{\infty}$ *are determined by (30)-(32) and*

1. *under* <u>*rationed money supply,*</u> *i.e. under a binding collateral constraint (27), by*

$$1/R_t^L = (1-\kappa_t)\,\beta c_t^{\sigma} E_t[c_{t+1}^{-\sigma}\pi_{t+1}^{-1}] + \kappa_t/R_t^m, \tag{33}$$

$$c_t = \kappa_t\left([mc_t/\left(1-\tau^n\right)]a_t\alpha n_t^{\alpha}/R_t^m\right) + \kappa_t^B\left(b_{t-1}/R_t^m\right)\pi_t^{-1} + m_{t-1}^H\pi_t^{-1}, \tag{34}$$

where $\{m_t^H,\ b_t\}_{t=0}^{\infty}$ *satisfy* $b_t + m_t^H = \Gamma\left(b_{t-1} + m_{t-1}^H\right)/\pi_t$ *and* $m_t^H\left(1+\Omega_t\right) = [m_{t-1}^H + \kappa_t^B b_{t-1}/R_t^m]\pi_t^{-1}$, *given* $\{a_t,\mu_t\}_{t=0}^{\infty}$, $\{R_t^m,\ \kappa_t,\ \kappa_t^B,\ \Omega_t\}_{t=0}^{\infty}$, $m_{-1}^H > 0$ *and* $b_{-1} > 0$.

2. *under* <u>*non-rationed money supply,*</u> *i.e. under a non-binding collateral constraint (27), by*

$$1/R_t^L = \beta c_t^{\sigma} E_t[c_{t+1}^{-\sigma}\pi_{t+1}^{-1}], \quad R_t^m = R_t^L, \tag{35}$$

given $\{a_t,\mu_t\}_{t=0}^{\infty}$ *and* $\{R_t^m\}_{t=0}^{\infty}$.

Proof. See Appendix B ■

Part 1 of Proposition 2 refers to the case where the collateral constraint (27) is binding, which requires the policy rate to be set according to $R_t^m < \overline{R}_t$, where $\overline{R}_t = 1/(\beta c_t^{\sigma} E_t[c_{t+1}^{-\sigma}\pi_{t+1}^{-1}])$ (see 28). The equilibrium conditions (33) and (34) show that the central bank can influence the allocation by changes in the three instruments R_t^m, κ_t^B, and κ_t. If a non-zero fraction of loans is accepted as collateral, $\kappa_t > 0$, the policy rate directly affects the loan rate (see 33), which alters the consumption-labor choice according to (31). The central bank can further induce changes in the demand for the cash good by tightening or expanding money supply with all instruments (see 34), given that the cash-in-advance constraint is then binding as well (see 26). Under money rationing the central bank can thus apply more than one instrument to affect the equilibrium allocation via the firms' borrowing costs as well as via aggregate demand, and can thereby enhance welfare when the first best allocation is not implementable with a single instrument.

If, in contrast, the policy rate equals the nominal marginal rate of intertemporal substitution $R_t^m = \overline{R}_t$, the collateral constraint (27) is slack and money supply is not (effectively) rationed.[18]

[18] Setting $R_t^m = \overline{R}_t$ is equivalent to the case where the central bank supplies money against assets which are abundantly available, like contingent claims $D_{i,t}$, which are issued by private agents. The policy rate can then not be lower than the marginal costs of money $\overline{R}_t$ in equilibrium, since abundantly available assets can be used to acquire more money, such that consumption increases and $\overline{R}_t$ decreases until $\overline{R}_t = R_t^m$.

The instruments κ_t and κ_t^B are then irrelevant for the allocation and the loan rate equals the policy rate (see part 2 of Proposition 2). The policy rate then affects consumption and expected inflation via the consumption Euler equation (35) and simultaneously via the consumption-leisure choice (see 31). Such a single instrument regime, can suffice for a central bank to conduct optimal policy if it faces just a single distortion. When first best can in fact be implemented by appropriately setting the policy rate, money rationing can be irrelevant or even not recommendable. This is summarized for the stylized case of perfect competition ($\varepsilon \to \infty$ and $\zeta_t \to \infty$) and perfectly flexible prices ($\phi = 0$) in the following Proposition.

Proposition 3 *Suppose that competition is perfect, prices are perfectly flexible, and $\tau^n = 0$. The allocation then equals first best if $R_t^L = 1$ and $\overline{R}_t = 1$ for $\gamma > 0$, requiring $R_t^m = 1$ and a non-rationed money supply.*

Proof. See Appendix B ■

Under perfectly flexible prices and perfect competition, which also implies that there are no cost-push shocks, the private sector behavior is only distorted by the liquidity constraints (1) and (3) that firms and households face. These distortions are undone when holding money is costless, which demands that the lending rate R_t^L equals one and, when households consume credit goods ($\gamma > 0$), that the nominal marginal rate of intertemporal substitution $\overline{R}_t$ also equals one. Both requirements together with (33) imply that the policy rate R_t^m equals $\overline{R}_t$, and, by (28), that money supply is not rationed. When households do not consume credit goods, $\gamma = 0$, the nominal marginal rate of intertemporal substitution does not need to equal one, such that first best can be implemented regardless whether money supply is rationed or not.[19] For less stylized cases where prices are sticky or cost-push shocks exist, a welfare maximizing monetary policy will, however, rely on money rationing, as it allows to address more than one distortion. Specifically, policies that aim at stabilizing prices or at neutralizing cost-push shocks and that simultaneously strive for low borrowing costs can be conducted in a more successful way under money rationing than under a single instrument regime. This will be shown in the subsequent Section.

4 Optimal monetary policy

In this Section, we examine optimal monetary policy under commitment, without the simplifying assumption of perfect competition. In the first part, we describe some main properties of the solution to the optimal policy problem. In the second part, we focus on the special case without credit goods, where money rationing enables the central bank to implement first best.

[19] For the case of a cashless economy with sticky prices, it is well established (see the discussion of divine coincidence in [3]) that a single instrument can also be sufficient to implement first best (if average mark-ups are eliminated). For this, the policy rate has to be set in a way that induces aggregate demand to be consistent with constant marginal costs, such that prices are never changed.

4.1 Optimal policy under money rationing

Throughout the subsequent analysis, we examine optimal monetary policy under the assumption that the central bank has access to a commitment technology that ensures that policy announcements are honored. It is well known that optimal policy under commitment may entail a time inconsistency when forward-looking equilibrium conditions – like (32) – serve as constraints to the policy problem. Following the related literature on optimal monetary policy (see, e.g., [23]), we disregard the issue of time-inconsistency and restrict our attention to time-invariant processes of the solution to the policy plan.[20]

In general, an optimal monetary policy will seek to minimize the effects of the distortions mentioned in Section 3. As demonstrated by Proposition 2, the central bank has more instruments available if it effectively rations money supply, i.e. if it sets the policy rate in a way that leads to a binding collateral constraint (27). To examine if the central bank actually chooses to ration money supply, we consider the collateral constraint (27) and the cash-in-advance constraint (3) as constraints to the policy problem. When deriving the optimal policy, we restrict our attention to cases where the constraints on the central bank instruments, $\kappa_t^B \in (0,1]$, $\kappa_t \in [0, \min\{1, \overline{\kappa}_t\}]$, and $R_t^m \geq 1$, are not binding. For example, the non-negativity constraint on interest rates imposes a relevant restriction on the policy rate, $R_t^m \geq 1$, which might be binding for large shocks or when the mean policy rate implied by the solution to the policy problem is close to unity. We therefore consider a production subsidy $\tau^n = 1 - (\varepsilon - 1)/(\varepsilon\mu)$ and standard deviations of shocks that allow to implement the optimal policy plan without violating these constraints on the instruments in the neighborhood of the steady state under the optimal policy.[21]

Examining the policy problem, it can be shown that the central bank will actually choose to ration money supply. Specifically, by setting the policy rate according to $R_t^m < \overline{R}_t$, the central bank can reduce the loan rate below $\overline{R}_t$ and offset mark-up shocks by state contingent adjustments of its additional instruments. Hence, the allocation under the optimal policy is independent of mark-up shocks, which is summarized as follows.

Proposition 4 *The optimal monetary policy under commitment is associated with money rationing, i.e.* $R_t^m < 1/(\beta c_t^\sigma E_t[c_{t+1}^{-\sigma}\pi_{t+1}^{-1}])$ *and* $\kappa_t > 0$*. The policy problem is then given by*

$$\max_{\{c_t, \widetilde{c}_t, n_t, \pi_t, s_t\}_{t=0}^\infty} E\sum_{t=0}^{\infty}\beta^t u(c_t, \widetilde{c}_t, n_t), \ s.t.\ (30),\ (32),\ s_t = \phi s_{t-1}\pi_t^\varepsilon + (1-\phi)^{\frac{1}{1-\varepsilon}}\left(1 - \phi\pi_t^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}}, \quad (36)$$

for $t = 0, ...\infty$ *and* $s_{-1} = 1$*, such that the solution to (36) is independent of cost-push shocks.*

Proof. See Appendix C ■

[20] Specifically, the optimality conditions that will be applied in the analysis can be interpreted as being part of a commitment plan derived and implemented in a timeless perspective (see, e.g., [26]).

[21] See Section 6 for a discussion of the limits to money rationing.

Since the loan rate R_t^L can be manipulated by setting R_t^m for $\kappa_t > 0$ according to (33), without simultaneously altering the nominal rate of intertemporal substitution (as in 35), the central bank can offset stochastic changes in the mark-up μ_t (see 31). This is in general not possible for the case of non-rationed money supply, where (31) is in general a binding constraint to the policy problem (see Appendix E). However, the remaining distortions cannot completely be removed under money rationing when households assign a positive value to credit goods, $\gamma > 0$. Specifically, avoiding a welfare reducing price dispersion requires stable goods prices, whereas avoidance of the inflation tax on the cash good calls for a deflation according to the Friedman rule (see 32). Since the choice of inflation cannot address both distortions simultaneously, first best cannot be achieved.[22] This property is summarized in the following Proposition.

Proposition 5 *Consider the policy problem in (36) for $\gamma > 0$. Under the optimal policy, prices are not stable in the long-run and the allocation differs from the first best allocation.*

Proof. See Appendix C ■

Given that prices are sticky, the central bank will not fully eliminate the inflation tax on cash goods induced by the households' cash-in-advance constraint (3). However, it can substantially reduce the distortion induced by the firms' liquidity constraint (1) under money rationing by setting the policy rate below the nominal marginal rate of intertemporal substitution, $R_t^m < \overline{R}_t$, and declaring a positive fraction of loans as eligible, $\kappa_t > 0$. This is apparently not possible under a non-rationed money supply, where the steady state values of the policy rate and the loan rate are identical and determined by π/β. Thus, the possibility to separate the policy rate from $\overline{R}_t$ allows the central bank to reduce the long-run effects of distortions as well as to stabilize the economy in the short-run in a more efficient way than under non-rationed money supply, which will be shown in Section 5.

4.2 A special case

In this Section, we examine the special case where there exists no cash-credit goods friction and average borrowing costs are eliminated, which is particularly useful for demonstrating how money rationing improves the central bank's ability to address distortions in the short-run. For this version of the model, we can derive a closed form solution to the policy problem. Specifically, we assume that households do not consume credit goods, $\gamma = 0$, such that the inflation tax does not distort the household decisions (as in 32). Nevertheless, the allocation between consumption and working time, which are both associated with liquidity requirements (see 1 and 3), is affected by the loan rate (see 31). Under money rationing, the central bank can then set the instruments

[22] It should be noted that the policy instrument κ_t^B might be non-stationary under the optimal policy. This can be the case when the growth rate of short-term treasuries Γ (see 23) deviates from the long-run inflation rate under the optimal policy (see Proposition 7).

such that welfare losses due to changes in mark-ups and due to price dispersion are avoided by manipulating the loan rate and aggregate demand in a way that eliminates the wedges in (31) and that implies constant goods prices. This however requires average mark-ups to be small and the average loan rate to be sufficiently large such that the zero interest rate bound is not hit. For this, we follow related studies on optimal monetary policy (see [20] and [6]) and consider a subsidy which corrects for the deterministic means of the mark-ups and the costs of borrowing:

$$\tau^n = 1 - [mc/\mu]/R^L, \tag{37}$$

where $\mu = \zeta/(\zeta-1)$ and $mc = (\varepsilon - 1)/\varepsilon$. When production is subsidized according to (37), monetary policy under money rationing can offset changes in mark-ups by induced changes in the loan rate (see 33), while money supply can be adjusted to implement a level of aggregate demand consistent with stable prices (see 34). Hence, the central bank can overcome the well-known trade-off between stabilizing inflation and closing the output gap by rationing money supply. This result is summarized in the following Proposition.

Proposition 6 *If households do not consume credit goods, $\gamma = 0$, and the production subsidy satisfies (37), the central bank can implement the first best allocation if it rations money supply.*

Proof. See Appendix C. ■

As stated in Proposition 6, the central bank can implement the first best allocation when it rations money supply, which endows the central bank with the two money supply instruments κ_t and κ_t^B in addition to the policy rate R_t^m. To see how the instruments are adjusted to implement first best, insert (37) in (31) to get

$$\chi n_t^{1+\eta-\alpha} c_t^\sigma = a_t \alpha \cdot \left[\frac{mc_t/mc}{\mu_t/\mu} \frac{R^L}{R_t^L} \right]. \tag{38}$$

A comparison of (30) and (38) for $\gamma = 0 \Rightarrow \widetilde{c}_t = 0$ with the corresponding conditions for the first best allocation (29) shows that $mc_t/mc = (\mu_t/\mu)\left(R_t^L/R^L\right)$ and $s_t = 1$ have to be satisfied for the allocation under the optimal policy to be identical with first best. The absence of price dispersion, $s_t = 1$, requires constant prices, $\pi_t = 1$, which is only consistent with optimal price setting when marginal costs are constant, $mc_t = mc$ (see proof of Proposition 7). Hence, the central bank can implement the first best allocation if and only if it sets its instruments such that

$$mc_t = mc \quad \text{and} \quad R_t^L/R^L = \mu/\mu_t. \tag{39}$$

Given that the cash-in-advance constraint and the collateral constraint are binding (see Proposition 4), the central bank can adjust R_t^m to induce the loan rate to satisfy $R_t^L/R^L = \mu/\mu_t$, and can set κ_t and κ_t^B in a state-contingent way that implies a consumption level which is consistent with $mc_t = mc$ (see Proposition 2). In particular, κ_t and κ_t^B can be adjusted according to (34) such that consumption equals $\overline{c}_t$ and – by (30) – working time equals $\overline{n}_t$, where $\overline{c}_t = \overline{c}(s_t, c_t^*)$ and $\overline{n}_t = \overline{n}(s_t, n_t^*)$.[23] Once $mc_t = mc$ holds, retailers will not change their prices, implying that there will be no price dispersion, $s_t = 1$, and the equilibrium allocation is identical to the first best allocation, $\overline{c}_t = c_t^*$ and $\overline{n}_t = n_t^*$. The following Proposition summarizes how the first best allocation can be implemented.

Proposition 7 *For $\gamma = 0$ and (37), the central bank implements price stability and the first best allocation if it sets $R_t^m \geq 1$, $\kappa_t \in [0, \min\{1, \overline{\kappa}_t\}]$, $\kappa_t^B \in (0, 1]$, and $\Omega_t \geq 0$ according to*

$$\kappa_t / R_t^m = \mu_t / \left(\mu R^L\right) - (1 - \kappa_t)\, \beta c_t^\sigma E_t[c_{t+1}^{-\sigma} \pi_{t+1}^{-1}] > 0, \tag{40}$$

$$\kappa_t^B b_{t-1} = R_t^m \pi_t \left[\overline{c}_t \left(1 - \kappa_t s_t mc_t \mu_t \alpha R_t^L / R_t^m\right) - m_{t-1}^H \pi_t^{-1}\right], \tag{41}$$

$R_t^m < 1/(\beta c_t^\sigma E_t[c_{t+1}^{-\sigma} \pi_{t+1}^{-1}])$, and $\lim_{t \to \infty}(\frac{1 + R^m \Omega_t / \kappa_{t+1}^B}{1 + \Omega_t} - \Gamma \frac{1 + R^m \Omega_{t-1} / \kappa_t^B}{1 + \Omega_{t-1}}) = 0$. If loans are not eligible, $\kappa_t = 0$, the first best allocation cannot be implemented.

Proof. See Appendix C. ■

As described in Proposition 7, optimal policy can implement the first best allocation by using the instruments R_t^m, κ_t, and κ_t^B. For this, the central bank has to accept at least some loans in open market operations, $\kappa_t > 0$, to satisfy (40). The conditions listed in Proposition 7 further imply that multiple combinations of the instruments are consistent with optimal policy. The central bank can directly manipulate the loan rate by setting the policy rate for a given κ_t (see 40),[24] and it can implement the desired consumption level $\overline{c}_t$ by adjusting κ_t^B or κ_t according to (41); the latter condition showing that a higher κ_t requires – ceteris paribus – a lower κ_t^B and vice versa. Under the optimal policy, the long-run inflation rate equals one, which can be implemented if the central bank adjusts κ_t^B or Ω_t in the long-run to off-set trends ($\Gamma > 1$ or $\Gamma < 1$) in the supply of government bonds, i.e. $\frac{1 + \Omega_t R^m / \kappa_{t+1}^B}{1 + \Omega_t} = \Gamma \frac{1 + \Omega_{t-1} R^m / \kappa_t^B}{1 + \Omega_{t-1}}$ for $t \to \infty$.[25]

If loans are not eligible, $\kappa_t = 0$, the loan rate R_t^L equals $\overline{R}_t$ (see 22) and is only affected by the policy rate through its equilibrium impact on consumption and inflation. In this case, the remaining instruments, i.e. R_t^m and κ_t^B, jointly affect the consumption level via (41), but they cannot simultaneously be used to off-set mark-up shocks. Put differently, changes in the policy rate do not directly affect firms' marginal costs if $\kappa_t = 0$, such that all instruments impact on the demand side and first best cannot be implemented. This result relates to the well-known trade-off faced by

[23] As shown in the proof of Proposition 7, the target values $\overline{c}_t$ and $\overline{n}_t$ are given by $\overline{c}_t = s_t^{-(1+\eta)/(\eta + \alpha\sigma + 1 - \alpha)} c_t^*$ and $\overline{n}_t = s_t^{(\sigma - 1)/(\eta + \alpha\sigma + 1 - \alpha)} n_t^*$, where the first best values c_t^* and n_t^* satisfy (29).

[24] This can be done in the simplest way when all loans are eligible, $\kappa_t = 1$, such that (40) reduces to $R_t^m / R^m = \mu / \mu_t$.

[25] For $\Gamma > 1$, the values of κ_t^B have to decline over time to implement a stationary sequence of inflation rates. If $\Gamma < 1$, the central bank has to increasingly supply money for treasuries under repos by letting the ratio of repos to money supplied outright Ω_t increase over time.

central banks conducting a conventional interest rate policy, which in our framework corresponds to the case where money is not rationed. The equilibrium is then characterized by (35), which together with (38) and (39) imply that under the first best allocation $\beta R^m (\mu/\mu_t) E_t (a_t/a_{t+1})^{\frac{\sigma+\eta\sigma}{\eta+\alpha\sigma+1-\alpha}}$ would have to equal 1. This is apparently impossible as long as either $R^m \neq \beta$, $\mu_t \neq \mu$, or $a_t \neq 1$. Hence, first best cannot be implemented when money supply is not rationed, since the single monetary policy instrument, R_t^m, cannot be set to simultaneously off-set mark-up shocks and to implement a consumption level consistent with constant prices.

Corollary 1 *When money supply is not rationed, the first best allocation cannot be implemented.*

5 Comparison with the case of non-rationed money supply

In this Section, we examine the long-run and short-run welfare gains of monetary policy under money rationing compared to a conventional (optimizing) monetary policy regime where the collateral constraint is slack, i.e. where the policy rate is restricted by $R_t^m = 1/(\beta c_t^\sigma E_t[c_{t+1}^{-\sigma}\pi_{t+1}^{-1}])$ (see 28), while we abstract from the simplifying assumptions ($\gamma = 0$ and $\tau^n = 1 - [mc/\mu]/R^L$) made in the previous Section.[26] The latter regime corresponds to the optimal monetary policy in standard sticky price models with transaction frictions, where money is supplied in a non-rationed way (see e.g. [17] or [20]). Given that the collateral constraint is slack, neither government bonds nor money holdings are relevant for the determination of the equilibrium allocation when the central bank sets the policy rate (see Proposition 2). Then, the set of constraints for the policy problem consists of $\tilde{Z}_t = \frac{\varepsilon}{\varepsilon-1} Z_{1,t}/Z_{2,t}$, where $Z_{1,t} = \gamma \widetilde{c}_t^{-\sigma} y_t mc_t + \phi\beta E_t \pi_{t+1}^{\varepsilon} Z_{1,t+1}$ and $Z_{2,t} = \gamma\widetilde{c}_t^{-\sigma} y_t + \phi\beta E_t \pi_{t+1}^{\varepsilon-1} Z_{2,t+1}$, $s_t = (1-\phi)\tilde{Z}_t^{-\varepsilon} + \phi s_{t-1}\pi_t^{\varepsilon}$, $1 = (1-\phi)(\tilde{Z}_t)^{1-\varepsilon} + \phi\pi_t^{\varepsilon-1}$, (30), (31), (32), and (35), where we eliminate the loan rate in (31), $\mu_t \chi n_t^{\eta+1-\alpha}/(mc_t \alpha a_t) = \beta E_t\left(c_{t+1}^{-\sigma}/\pi_{t+1}\right)$.[27]

For the numerical analysis of the *optimal* policy regime with money rationing (labeled with *opt*) and of the policy regime with non-rationed money supply (*std*), which accords to a *standard* optimal policy regime, we calibrate the model using standard parameter values as far as possible. The labor income share equals $\alpha = 0.66$, the substitution elasticity for intermediate goods $\varepsilon = 10$, steady state working time $n = 0.33$, the discount rate $\beta = 0.99$, the fraction of non-optimally price adjusting firms $\phi = 0.8$, and the elasticities of the utility function equal $\sigma = 2$ and $\eta = 0$. For the utility weight of credit goods γ, we apply a benchmark value of $\gamma = 1$, implying that both types of consumption goods are treated in an identical way. We further assume the log of the productivity level and the log of the wage mark-up $\mu_t = \zeta_t/(\zeta_t - 1)$ to be generated by the (AR1) processes $a_t = a^{1-\rho} a_{t-1}^{\rho} \exp(\varepsilon_t)$ and $\mu_t = \mu^{1-\rho_\mu}\mu_{t-1}^{\rho_\mu}\exp(\varepsilon_t^{\mu})$, where $E_{t-1}\varepsilon_t^{(\mu)} = 0$, $a = 1$,

[26] In Appendix F, we show that money rationing also lead to long-run and short-run welfare gains in an alternative model version, where heterogenous households borrow/lend for consumption purposes and consumption loans are eligible for open market operations.

[27] The policy problem is summarized in Appendix E.

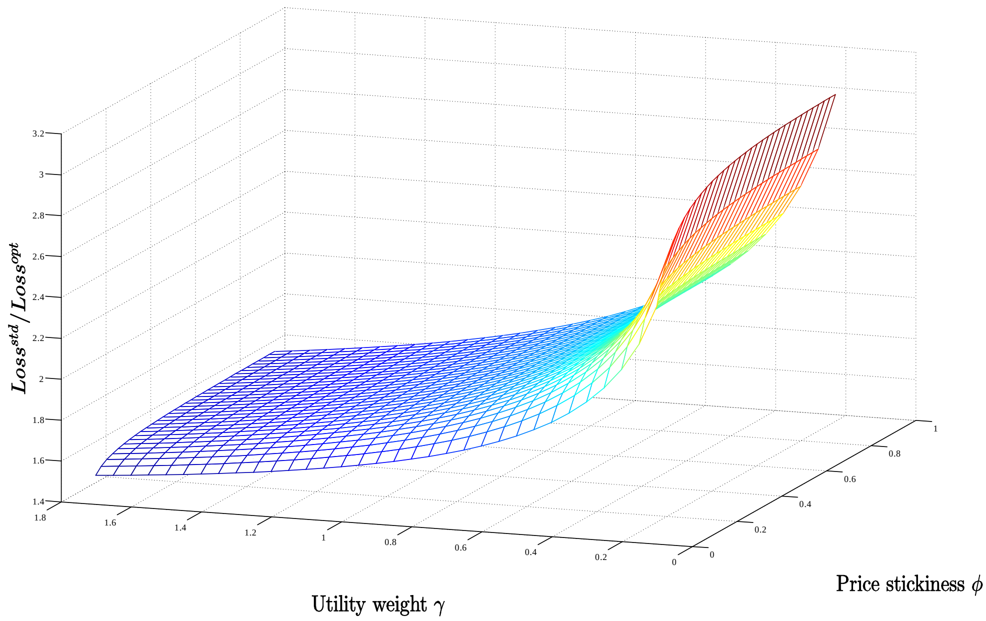

Figure 1: Steady state welfare gain of money rationing

$\mu = 10/9$, autocorrelation coefficients $\rho_{(\mu)}$ equal 0.9, and standard deviations of $\varepsilon_t^{(\mu)}$ equal 0.005 (see e.g. [22]).

We first compare steady state welfare under the optimal monetary policy as described in Section 4.1 with steady state welfare under an else optimal policy regime with non-rationed money supply. For the computation of the solutions to these policies we assume that the production subsidy eliminates average mark-up distortions, $\tau^n = 1 - [(\varepsilon - 1)/(\varepsilon\mu)]$.[28] Welfare is measured by using the households' objective, $V^x = E\sum_{t=0}^{\infty}\beta^t u(c_t^x, \widetilde{c}_t^x, n_t^x)$ where $x \in \{*, opt, std\}$, under the assumption that the initial values are identical with the corresponding steady state values. Deviations from the welfare value under the first best $(*)$ are then measured as permanent consumption values that compensate for the welfare loss under alternative policy regimes, $Loss^x = c_{perm}^x - c_{perm}^*$, where $c_{perm}^x = ((1-\beta)(1-\sigma)V^x + 1)^{1/(1-\sigma)}$.

Figure 1 presents the long-run welfare gain of money rationing, i.e. the ratio of the welfare loss under the optimal policy regime without money rationing to the welfare loss under optimal policy with money rationing, $Loss^{std}/Loss^{opt}$, for values of the utility weight of credit goods $\gamma \in [0.15, 1.75]$ and for values of the degree of price stickiness $\phi \in [0.075, 0.875]$. It shows that the welfare loss under a regime with non-rationed money supply exceeds the welfare loss under an optimal policy regime by a factor that is larger than 1.4. This ratio tends to increase monotonically (up to 3) with a higher degree of price stickiness ϕ, which aggravates the welfare costs of price

[28] The full sets of steady state conditions are given in Appendix C and in Appendix E.

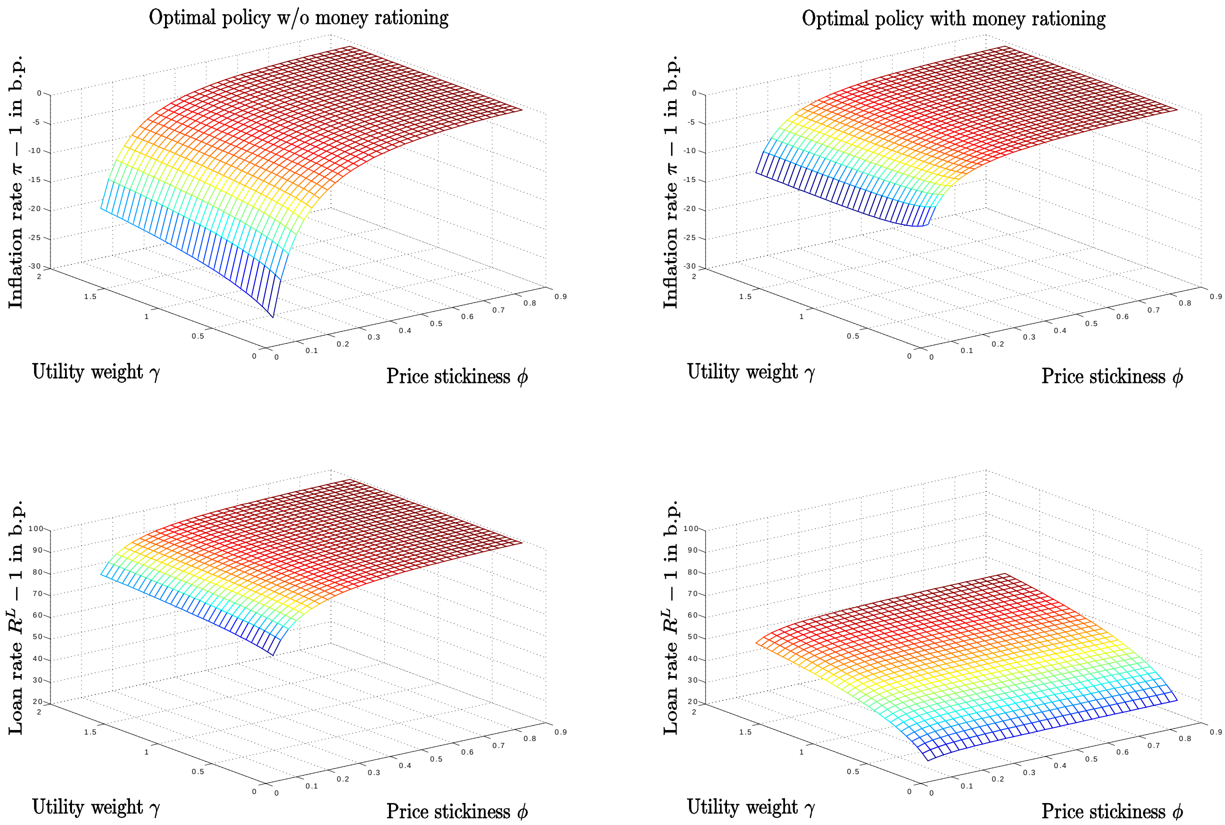

Figure 2: Steady state inflation rates and loan rates (in basis points) under rationed and non-rationed money supply

instability, and with a smaller utility weight γ, which reduces the welfare costs of the inflation tax on cash goods (see Section 4.2 for the special case $\gamma = 0$). The reason for this long-run welfare gain of money rationing is the separation of the policy rate and the nominal rate of intertemporal substitution, which equals π/β in the steady state. The inflation rates under both regimes (*std*, *opt*), which are shown in first row of Figure 2, are close to one for higher degrees of price stickiness, $\phi \geq 0.5$, which is due to the central bank's aim to avoid welfare costs of price dispersion. For lower values of ϕ, the central bank is increasingly willing to lower the inflation tax on credit goods, such that inflation rates are smaller (which accords to [23]). The deviation from price stability is thereby more pronounced when money is not rationed. According to the left panel in the second row of Figure 2, the loan rate under non-rationed money supply actually reflects the pattern of the inflation rate (where the axis is rescaled for convenience). The striking difference to the optimal policy with money rationing can be seen from inspecting the right panel in the second row, showing that the borrowing rate, which distorts firms' labor demand, is lowered by 1.6–2.8 % in terms of annualized rates. This reduction in the loan rate is made possible by a policy rate set below π/β and by eligibility of corporate loans, $\kappa > 0$ (see 33).

To demonstrate short-run differences between the monetary policy regimes with money ra-

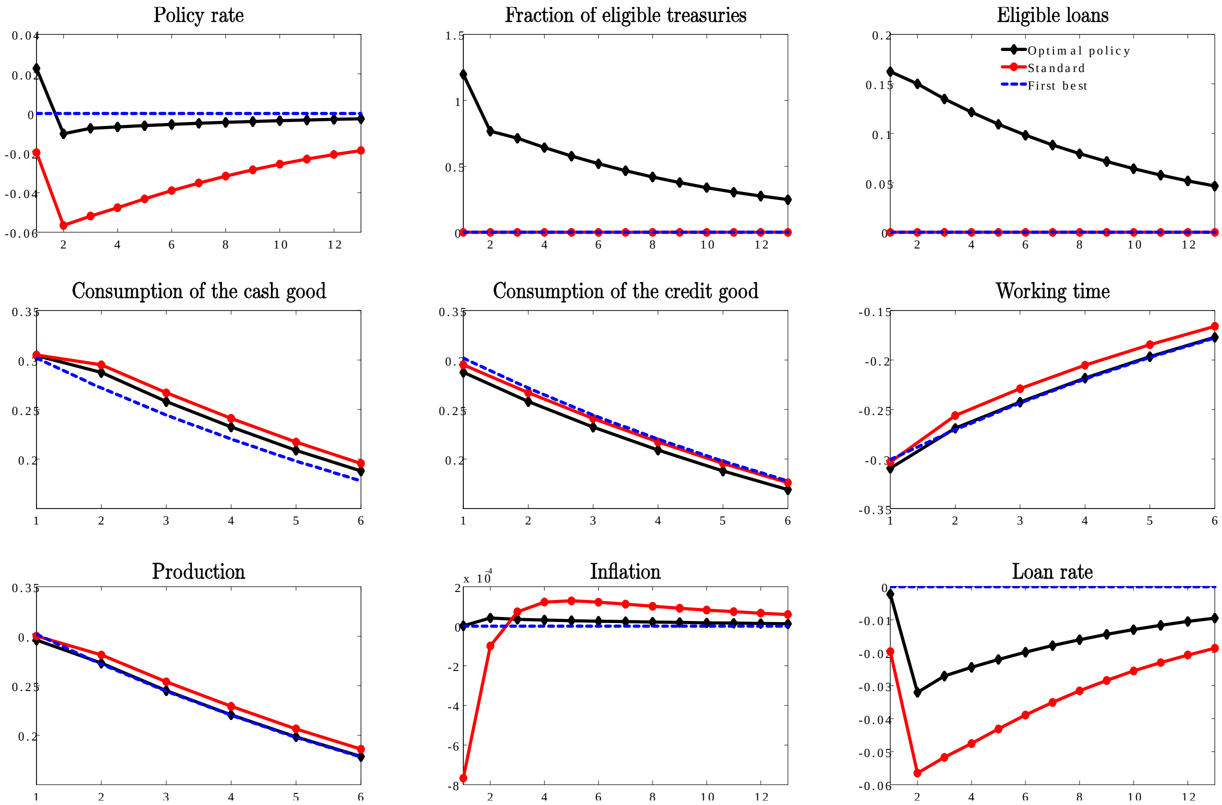

Figure 3: Responses (in % dev. from the steady states) to a productivity shock

tioning and without money rationing, we compute impulse responses to aggregate shocks. As stated in Proposition 7, an optimal policy regime has to satisfy $\kappa_t > 0$. Here, we keep κ_t constant and set it equal to 0.55 (implying a steady state ratio of eligible loans to output equal to 1/3), while the other instruments, κ_t^B and R_t^m, are adjusted in a state contingent way according to (40) and (41). We further set the repo share Ω_t equal to 3/4, which implies the steady state values of κ_t^B and R_t^m to equal 0.75 and 1.004 (in annualized terms). To simplify the analysis, we further assume that Γ equals the prevailing long-run inflation rate, which allows to abstract from long-run adjustments in κ_t^B or Ω_t (see Proposition 7). Figures 3 and 4 show impulse responses to aggregate shocks (a_t and μ_t) under different scenarios, i.e. the optimal policy (solid black line with diamonds), the optimal policy under non-rationed money supply (red solid circled line), and first best (blue dashed line). Overall, they indicate that the economy can be stabilized in a more successful way when money is rationed, which is confirmed by a short-run welfare gain of $(Loss_{total}^{std} - Loss^{std})/(Loss_{total}^{opt} - Loss^{opt}) = 3.15$, where $Loss_{total}^{std}$ and $Loss_{total}^{opt}$ are total welfare losses (defined analogous to the steady state welfare losses $Loss^x$).[29] Notably, the short-run

[29] The total welfare losses $Loss_{total}^{std}$ and $Loss_{total}^{opt}$ are measured by the stochastic means of the values V_t^{std} and V_t^{opt} computed by applying a second order approximation – as implemented in *dynare* 4.2.2 – of the solutions to the policy plans *std* and *opt*.

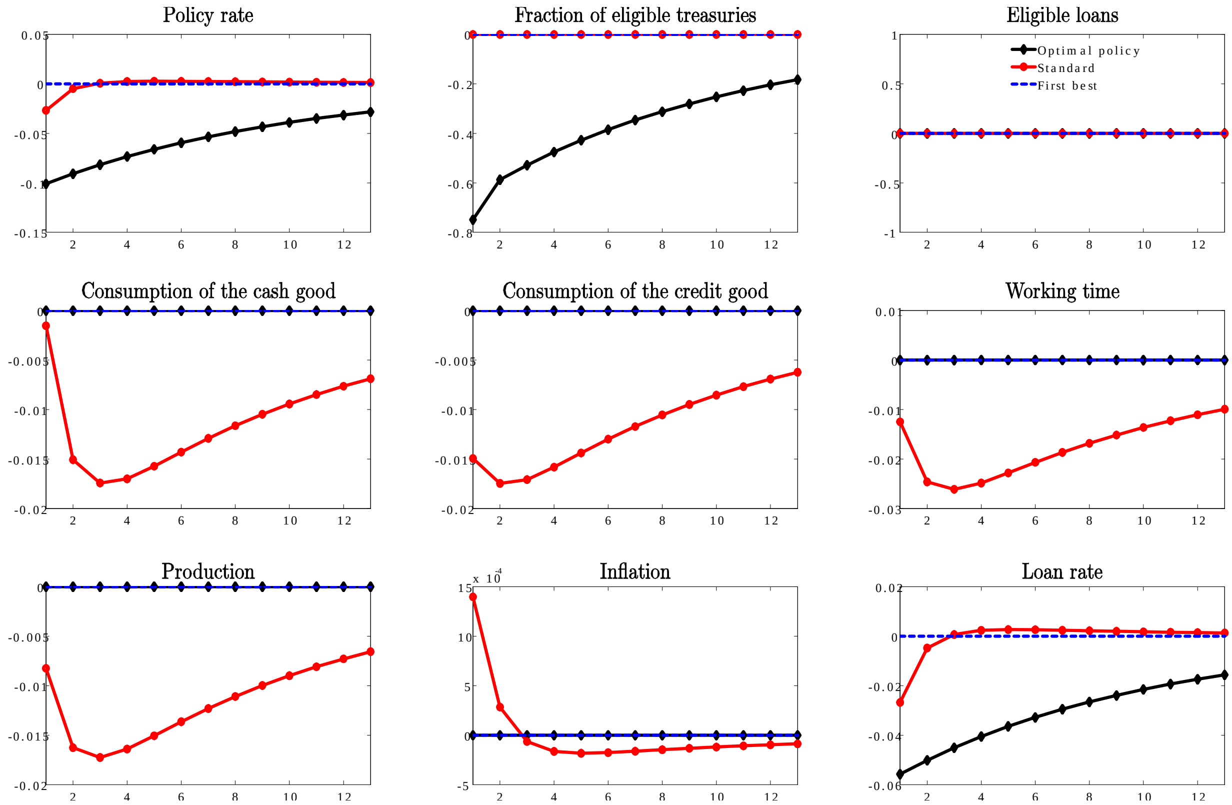

Figure 4: Responses (in % dev. from the steady states) to a cost push shock

stabilization gain is considerably smaller (1.2) when only productivity shocks are considered.

Figure 3 shows the impulse responses to a positive productivity shock, which tends to lower firms' total marginal costs and thereby the inflation rate (see 8 and 31). Overall, the allocations under the three scenarios respond in a similar way and exhibit only small differences. Working time and the production under the optimal policy and under the first best are almost identical, while consumption of the credit good under the non-rationed money supply regime comes closer to the first best counterpart. The response of consumption of the cash good under the optimal policy regime lies between the other scenarios. When money is non-rationed, the central bank lowers the policy rate (that here equals the loan rate) in response to higher productivity. The central bank thereby further stimulates aggregate demand (which is reflected by the output response), such that the fall in total marginal costs and thus inflation is mitigated, while both, the costs of borrowing and the inflation tax, are reduced. In contrast, under the optimal policy regime (with rationed money supply), the central bank initially raises the policy rate, which tends to stabilize firms' total marginal costs. At the same time, the amount of money supplied against eligible assets increases due to an increase of loans and a larger amount of money supplied against treasuries κ_t^B, which offsets the impact of the policy rate response on aggregate demand. This strategy allows to stabilize inflation in a more successful way than under the policy regime without money rationing.

Figure 4 shows impulse responses to a cost-push shock, i.e. a wage mark-up shock, which does neither affect the first best allocation (see Proposition 4) nor the allocation under optimal policy (with rationed money supply). Under a conventional policy regime (with non-rationed money supply), which is well-established not to be able to simultaneously stabilize prices and the allocation, the central bank slightly lowers the policy rate, which tends to stimulate aggregate demand but fails to fully stabilize prices. Inflation then increases on impact, while consumption and working time decrease. Under the optimal policy regime (with rationed money supply), the central bank lowers the policy rate and thereby the loan rate in a much more pronounced way to off-set the cost-increasing effect of the mark-up shock. To stabilize the allocation at this lower policy rate, which tends to stimulate real activity, the central bank reduces the fraction of eligible treasuries. Hence, by simultaneously adjusting the amount of rationed money and the policy rate, the effects of cost-push shocks can be neutralized, such that neither the allocation nor prices are affected.

6 Discussion

In this Section, we discuss the limits to optimal policy under money rationing and argue that collateralized central bank lending, as modelled in here, is not equivalent to direct central bank lending.

In the previous analysis, we have shown how a central bank can enhance welfare by money rationing. However, the scope of this strategy is limited by the restrictions on the policy instruments (like $\kappa_t^{(B)} \geq 0$ or $R_t^m \geq 1$) and, in particular, by the requirement that the collateral constraint (27) must be binding. As discussed in Section 2.5, this relies on the policy rate to be smaller than the nominal marginal rate of intertemporal substitution (see 28). Throughout the analysis, the instruments have in fact always been adjusted in a way that is consistent with these requirements, while this cannot be guaranteed for any size of aggregate shocks. For example, when the central bank keeps the fraction of eligible loans constant, an increase of the policy rate in response to a large productivity shock (see Figure 3) might render money rationing impossible due to a slack collateral constraint. The central bank is nevertheless able to conduct optimal policy by adjusting the fraction of eligible loans κ_t instead of (or with) the policy rate, which both affect the loan rate via (33).[30] The central bank can therefore increase the scope of this policy by applying all available instruments in a suited way. This possibility is particularly relevant at the zero lower bound, where stimulation of real activity by a reduction in the policy rate is not possible.[31]

To understand how collateralized central bank lending differs from direct lending, suppose that

[30] For example, a strong reduction of κ_t, which is consistent with a binding collateral constraint, can serve as a substitute for an increase in the policy rate.

[31] The case of a binding zero lower bound for the policy rate is examined by [16] in a closely related framework.

the central bank supplies loans directly to firms at the policy rate (rather than lends to households against loans as collateral). When the central bank lends only to a fraction $\kappa < 1$ of firms, firms who borrow from households will face higher costs of borrowing than firms receiving loans directly from the central bank, if the policy rate is below the nominal marginal rate of intertemporal substitution, $R_t^m < \overline{R}_t$.[32] As a consequence, a fraction κ of firms which can borrow at the policy rate will hire more labor and produce more than firms borrowing from households at the rate $R_t^L = \overline{R}_t$ (see 35). Firms would then produce at different levels, such that direct lending does not only affect credit conditions but also distorts the allocation of resources. In our set-up, the central bank however randomly selects eligible assets and thus avoids firms facing different costs of borrowing, such that firms are identical when loan contracts are signed. Hence, the commonly raised critique that deviating from a "Treasuries-only" collateral policy distorts the credit allocation applies to direct central bank lending (see [15]), but not to the non-discriminating type of collateralized lending as specified in this paper. Under direct lending, direct costs of operations other than open market operations in terms of treasuries can further be justified by credit origination by the central bank, as argued by [10]. Under collateralized lending, there is however no fundamental difference between supplying money temporarily against treasuries or against loans, when both are eligible.

7 Conclusion

In contrast to the predominant view on monetary policy, a central bank can simultaneously control the price and the amount of money, which requires to ration money supply in open market operations. This paper shows that central banks can enhance welfare via money rationing compared to a conventional policy regime where the central bank supplies money to satiate money demand at a given short-run nominal interest rate. Controlling the price as well as the quantity of money under money supply rationing allows to affect the private sector behavior via more than an interest rate channel. Under money rationing, the policy rate is decoupled from the nominal marginal rate of intertemporal substitution, which can lead to long-run welfare gains by implementing interest rates on eligible assets below levels implied by the Fisher equation. We further show that monetary policy can overcome the well-known short-run trade-off between stabilizing prices and closing output-gaps, which is – in the literature on optimal monetary policy under sticky prices – seen as the main task of a central bank. We expect the additional monetary policy instruments under money rationing also to be beneficial when frictions other than those considered in this paper are present.

[32] Such a policy would be comparable to the case where the central bank discriminates between firms when it accepts loans as collateral, such that households demand different loan rates depending on whether debt issued by a specific firm is eligible or not.

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Appendix

A Competitive Equilibrium

A competitive equilibrium can be defined as follows:

Definition 1 *A competitive equilibrium is a set of sequences* $\{c_t,\ \widetilde{c}_t,\ n_t,\ m_t^H,\ m_t^L,\ b_t,\ b_t^T,\ l_t,\ w_t,\ mc_t,\ \tilde{Z}_t,\ s_t,\ \pi_t,\ R_t^L\}_{t=0}^{\infty}$ *satisfying*

$$\mu_t \chi n_t^{\eta} = w_t c_t^{-\sigma}, \tag{A.1}$$

$$1/R_t^L = (1-\kappa_t)\, c_t^{\sigma} \beta E_t \left[c_{t+1}^{-\sigma}/\pi_{t+1}\right] + \kappa_t / R_t^m, \tag{A.2}$$

$$\gamma \widetilde{c}_t^{-\sigma} = \beta E_t \left[c_{t+1}^{-\sigma}/\pi_{t+1}\right], \tag{A.3}$$

$$c_t = (1+\Omega_t) m_t^H + m_t^L, \text{ if } \psi_t = c_t^{-\sigma}[1 - c_t^{\sigma}\beta E_t \left[c_{t+1}^{-\sigma}/\pi_{t+1}\right]] > 0, \tag{A.4}$$

$$\text{or } c_t \leq (1+\Omega_t) m_t^H + m_t^L, \text{ if } \psi_t = 0,$$

$$\kappa_t^B b_{t-1}/\left(R_t^m \pi_t\right) = (1+\Omega_t) m_t^H - m_{t-1}^H \pi_t^{-1}, \text{ if } \eta_t = c_t^{-\sigma}[(1/R_t^m) - c_t^{\sigma}\beta E_t \left[c_{t+1}^{-\sigma}/\pi_{t+1}\right] > 0, \tag{A.5}$$

$$\text{or } \kappa_t^B b_{t-1}/\left(R_t^m \pi_t\right) \geq (1+\Omega_t) m_t^H - m_{t-1}^H \pi_t^{-1}, \text{ if } \eta_t = 0,$$

$$\kappa_t l_t / R_t^m = m_t^L, \text{ if } \eta_t > 0, \text{ or } \kappa_t l_t / R_t^m \geq m_t^L, \text{ if } \eta_t = 0, \tag{A.6}$$

$$b_t = b_t^T - m_t^H, \tag{A.7}$$

$$mc_t a_t \alpha n_t^{\alpha-1} = (1-\tau^n)\, w_t R_t^L, \tag{A.8}$$

$$l_t / R_t^L = w_t n_t, \text{ if } R_t^L > 1, \text{ or } l_t / R_t^L \geq w_t n_t, \text{ if } R_t^L = 1, \tag{A.9}$$

$$\tilde{Z}_t\,(\varepsilon-1)/\varepsilon = Z_{1,t}/Z_{2,t}, \tag{A.10}$$

$$\text{where } Z_{1,t} = \gamma \widetilde{c}_t^{-\sigma} y_t mc_t + \phi\beta E_t \pi_{t+1}^{\varepsilon} Z_{1,t+1} \text{ and } Z_{2,t} = \gamma \widetilde{c}_t^{-\sigma} y_t + \phi\beta E_t \pi_{t+1}^{\varepsilon-1} Z_{2,t+1},$$

$$1 = (1-\phi)(\tilde{Z}_t)^{1-\varepsilon} + \phi \pi_t^{\varepsilon-1}, \tag{A.11}$$

$$b_t^T = \Gamma b_{t-1}^T / \pi_t, \tag{A.12}$$

$$a_t n_t^{\alpha} / s_t = c_t + \widetilde{c}_t, \tag{A.13}$$

$$s_t = (1-\phi)\tilde{Z}_t^{-\varepsilon} + \phi s_{t-1} \pi_t^{\varepsilon}, \tag{A.14}$$

the transversality conditions, a subsidy τ^n*, a monetary policy setting* $\{R_t^m \geq 1,\ \kappa_t^B \in (0,1],\ \kappa_t \in [0, \max\{1, \overline{\kappa}_t\}],\ \Omega_t \geq 0\}_{t=0}^{\infty}$*, given* $\{a_t, \mu_t\}_{t=0}^{\infty}$*,* $b_{-1} > 0$*,* $b_{-1}^T > 0$*,* $m_{-1}^H > 0$*, and* $s_{-1} = 1$*.*

When the policy rate R_t^m equals $1/(c_t^{\sigma}\beta E_t \left[c_{t+1}^{-\sigma}/\pi_{t+1}\right])$, $\eta_t = 0$ holds (see 28) and money supply is not effectively rationed (see A.5 and A.6). Then, the set of equilibrium conditions can be reduced to a conventional sticky price model with transaction frictions (see [17] or [20]) and the competitive equilibrium can be defined as follows.

Definition 2 *When money supply is not rationed,* $\eta_t = 0$*, a competitive equilibrium is a set of sequences* $\{c_t,\ \widetilde{c}_t,\ n_t,\ l_t,\ w_t,\ mc_t,\ \tilde{Z}_t,\ s_t,\ \pi_t,\ R_t^L\}_{t=0}^{\infty}$ *satisfying (A.1), (A.3), (A.8)-(A.11), (A.13), (A.14),* $c_t^{-\sigma} = \beta R_t^L E_t[c_{t+1}^{-\sigma}\pi_{t+1}^{-1}]$*,* $R_t^L = R_t^m$*, the transversality conditions, a monetary policy setting* $\{R_t^m \geq 1\}_{t=0}^{\infty}$*, given* $\{a_t, \mu_t\}_{t=0}^{\infty}$ *and* $s_{-1} = 1$*.*

The sum of real balances can then residually be determined by $c_t = (1+\Omega_t) m_t^H + m_t^L$ if $R_t^m > 1$, while there are infinitely many sequences for b_t and b_t^T that are consistent with a competitive equilibrium as given in Definition 2.

B Appendix to the Role of Money Rationing

Proof of Proposition 1. To identify the first best allocation, we consider a social planer who solves $\max E\sum_i \sum_{t=0}^{\infty} \beta^t u(c_{i,t}, \widetilde{c}_{i,t}, n_{i,t})$, s.t. 1.) $a_t \int_0^1 n_{j,t}^{\alpha} dj = \int_0^1 y_{k,t} dk$, 2.) $\int_0^1 n_{j,t} dj = \int_0^1 n_{i,t} di$, and 3.) $\int_0^1 y_{k,t}^{\frac{\varepsilon-1}{\varepsilon}} dk = (\int_0^1 c_{i,t} di + \int_0^1 \widetilde{c}_{i,t} di)^{\frac{\varepsilon-1}{\varepsilon}}$, where the last condition stems from combining $y_t^{\frac{\varepsilon-1}{\varepsilon}} = \int_0^1 y_{k,t}^{\frac{\varepsilon-1}{\varepsilon}} dk$ and $y_t = \int_0^1 c_{i,t} di + \int_0^1 \widetilde{c}_{i,t} di$. The first order conditions for $c_{i,t}$, $\widetilde{c}_{i,t}$, $n_{i,t}$, $n_{j,t}$, and $y_{k,t}$ are given by $\chi n_{i,t}^{\eta} = \lambda_{2,t}$, $\lambda_{1,t} a_t \alpha n_{j,t}^{\alpha-1} = \lambda_{2,t}$, $\lambda_{3,t} \frac{\varepsilon-1}{\varepsilon} y_{k,t}^{\frac{\varepsilon-1}{\varepsilon}-1} = \lambda_{1,t}$,

$$c_{i,t}^{-\sigma} = \lambda_{3,t} \frac{\varepsilon-1}{\varepsilon} \left(\int_0^1 c_{i,t} di + \int_0^1 \widetilde{c}_{i,t} di \right)^{\frac{\varepsilon-1}{\varepsilon}-1}, \quad \gamma \widetilde{c}_{i,t}^{-\sigma} = \lambda_{3,t} \frac{\varepsilon-1}{\varepsilon} \left(\int_0^1 c_{i,t} di + \int_0^1 \widetilde{c}_{i,t} di \right)^{\frac{\varepsilon-1}{\varepsilon}-1},$$

where the $\lambda's$ denote the multipliers on the three constraints. The RHSs of these conditions imply that choices for individual households, firms and retailers (i, j, and k) are identical, such that $\int_0^1 y_{k,t}^{\frac{\varepsilon-1}{\varepsilon}} dk = y_{k,t}^{\frac{\varepsilon-1}{\varepsilon}}$ and $c_{i,t} = c_t = \int_0^1 c_{i,t} di$, $\widetilde{c}_{i,t} = \widetilde{c}_t = \int_0^1 \widetilde{c}_{i,t} di$, $n_{i,t} = n_t = \int_0^1 n_{i,t} di$, and $n_{j,t} = n_t = \int_0^1 n_{j,t} di$. Eliminating the multipliers, yields $\chi n_t^{\eta} c_t^{\sigma} = a_t \alpha n_t^{\alpha-1}$, $\gamma \widetilde{c}_t^{-\sigma} = c_t^{-\sigma}$, and $\widetilde{c}_t + c_t = a_t n_t^{\alpha}$. Substituting out $\widetilde{c}_t$ with $\widetilde{c}_t = c_t \gamma^{1/\sigma}$, we get $n_t^{\eta+1-\alpha} = c_t^{-\sigma} a_t (\alpha/\chi)$ and $a_t n_t^{\alpha} = c_t (1+\gamma^{1/\sigma})$, and substituting out c_t with $c_t = a_t n_t^{\alpha} / (1+\gamma^{1/\sigma})$, gives $n_t = [a_t^{1-\sigma} (1+\gamma^{1/\sigma})^{\sigma} (\alpha/\chi)]^{\frac{1}{\eta+1-\alpha+\sigma\alpha}}$, which establishes the claims made in the Proposition. ■

Proof of Proposition 2. Suppose that the policy rate satisfies $R_t^m < 1/(c_t^{\sigma} \beta E_t[c_{t+1}^{-\sigma} \pi_{t+1}^{-1}]$. Then, $\eta_t > 0$ (see 28) and $\psi_t > 0$ (see 26) hold, such that (A.4)-(A.6) are binding. Hence, $\kappa_t^B b_{t-1} / (R_t^m \pi_t) = c_t - \kappa_t l_t / R_t^m - m_{t-1}^H \pi_t^{-1}$ has to be satisfied in equilibrium, which can by substituting out loans and the real wage rate with (A.8) and (A.9) be rewritten as

$$\kappa_t^B b_{t-1} / (R_t^m \pi_t) = c_t - \kappa_t (mc_t a_t \alpha n_t^{\alpha} / [(1-\tau^n) R_t^m]) - m_{t-1}^H \pi_t^{-1}. \tag{B.1}$$

Further substitute out w_t in (A.1) by (A.8) and b_t^T in (A.12) with (A.7). For given sequences $\{s_t, \pi_t\}_{t=0}^{\infty}$, the set of sequences $\{c_t, \widetilde{c}_t, n_t, mc_t, R_t^L\}_{t=0}^{\infty}$ can then be determined by (A.2), (A.3), (A.13), (B.1), and

$$\chi n_t^{\eta} c_t^{\sigma} = a_t \alpha n_t^{\alpha-1} [mc_t / (\mu_t (1-\tau^n))] / R_t^L, \tag{B.2}$$

where $\{m_t^H, b_t\}_{t=0}^{\infty}$ satisfy $m_t^H = \pi_t^{-1} [m_{t-1}^H + \kappa_t^B b_{t-1} / R_t^m] / (1+\Omega_t)$ and $b_t + m_t^H = \Gamma (b_{t-1} + m_{t-1}^H) / \pi_t$, given $\{a_t, \mu_t\}_{t=0}^{\infty}$, $\{R_t^m, \kappa_t, \kappa_t^B, \Omega_t\}_{t=0}^{\infty}$, $m_{-1}^H > 0$ and $b_{-1} > 0$.

Now suppose that $R_t^m = 1/(\beta E_t[c_{t+1}^{-\sigma} c_t^{\sigma} \pi_{t+1}^{-1}] \Rightarrow \eta_t = 0$ (see 28), such that (A.5) and (A.6) are slack, and (A.2) reduces to

$$1/R_t^L = \beta c_t^{\sigma} E_t[c_{t+1}^{-\sigma} \pi_{t+1}^{-1}]. \tag{B.3}$$

Hence, for given sequences $\{s_t, \pi_t\}_{t=0}^{\infty}$, the set of sequences $\{c_t, \widetilde{c}_t, n_t, mc_t, R_t^L\}_{t=0}^{\infty}$ are for $\eta_t = 0$ determined by (A.3), (A.13), (B.2), (B.3), and $R_t^L = R_t^m$, given $\{a_t, \mu_t\}_{t=0}^{\infty}$ and $\{R_t^m\}_{t=0}^{\infty}$. The sum of real balances can then residually be determined by (A.4) given $\{\Omega_t\}_{t=0}^{\infty}$ if $R_t^m > 1$. ■

Proof of Proposition 3. Suppose that competition is perfect, $\varepsilon \to \infty$ and $\zeta_t \to \infty$, prices are perfectly flexible, $\phi = 0$, and $\tau^n = 0$. Then, (8), (A.11), and (A.14) imply $mc_t = 1$, and $s_t = 1$, such that the conditions (30)-(32) reduce to

$$a_t n_t^{\alpha} = (c_t + \widetilde{c}_t), \; \chi n_t^{1+\eta-\alpha} c_t^{\sigma} = a_t \alpha (1/R_t^L), \; \gamma \widetilde{c}_t^{-\sigma} = c_t^{-\sigma} / \overline{R}_t, \tag{B.4}$$

where we used $\overline{R}_t = 1/(\beta c_t^{\sigma} E_t[c_{t+1}^{-\sigma} \pi_{t+1}^{-1}])$. According to (B.4), implementation of the first best (29) requires $R_t^L = 1$ and $\overline{R}_t = 1$ for $\gamma > 0$. Inserting these values in (33), implies the policy rate to satisfy $R_t^m = 1$. Then, $\eta_t = 0$ (see 28) and the collateral constraint (27) is not binding. ■

C Appendix to Optimal Monetary Policy

In this Appendix, we examine optimal monetary policy under commitment, where we consider that the central bank supplies money against eligible assets. The production subsidy is assumed to satisfy $\tau^n = 1 - (\varepsilon - 1)/(\varepsilon \mu)$, such that average distortions due to price and wage mark-ups are neutralized. We restrict our attention to the case where the constraints on the choice of the policy instruments, $R_t^m \geq 1$, $\kappa_t^B \in (0, 1]$, $\kappa_t \in [0, \max\{1, \overline{\kappa}_t\}]$, and $\Omega_t \geq 0$, are not binding, which is confirmed for the solution to the optimal policy plan. The central bank chooses an optimal plan for all periods $t \geq 0$, i.e. it maximizes household welfare (9) by choosing a set of sequences $\{c_t$, $\widetilde{c}_t$, n_t, m_t^H, m_t^L, b_t, b_t^T, l_t, w_t, mc_t, $\tilde{Z}_t$, s_t, π_t, $R_t^L\}_{t=0}^{\infty}$ and the instruments $\{\kappa_t^B, \kappa_t, R_t^m, \Omega_t\}_{t=0}^{\infty}$ subject to the set of equilibrium conditions (A.1)-(A.14). Substituting out w_t in (A.1) and (A.9) with (A.8) and m_t^L in (A.4) with (A.6), we can summarize the policy problem as follows:

$$\begin{aligned}
&\max_{\{c_t, \widetilde{c}_t, n_t, m_t^H, b_t, b_t^T, l_t, mc_t, \tilde{Z}_t, Z_{1,t}, Z_{2,t} s_t, \pi_t, R_t^L, \kappa_t^B, \kappa_t, R_t^m\}_{t=0}^{\infty}} \; \min_{\{\theta_{1,t}, \ldots \theta_{14,t}\}_{t=0}^{\infty}} \qquad (C.1)\\
&E \sum_{t=0}^{\infty} \beta^t \Big\{ \left[(c_t^{1-\sigma} - 1)(1-\sigma)^{-1} + \gamma (\widetilde{c}_t^{1-\sigma} - 1)(1-\sigma)^{-1} - \chi n_t^{1+\eta} (1+\eta)^{-1} \right] \\
&+ \theta_{1,t} \left[\mu_t (1 - \tau^n) \chi n_t^{\eta} c_t^{\sigma} - mc_t a_t \alpha n_t^{\alpha-1} / R_t^L \right] + \theta_{2,t} \left[\gamma \widetilde{c}_t^{-\sigma} - \beta E_t \left[c_{t+1}^{-\sigma} / \pi_{t+1} \right] \right] \\
&+ \theta_{3,t} \left[1/R_t^L - (1 - \kappa_t) c_t^{\sigma} \gamma \widetilde{c}_t^{-\sigma} - \kappa_t / R_t^m \right] + \theta_{4,t} \left[(1 + \Omega_t) m_t^H + \kappa_t l_t / R_t^m - c_t \right] \\
&+ \theta_{5,t} \left[b_t - b_t^T + m_t^H \right] + \theta_{6,t} \left[l_t - mc_t a_t \alpha n_t^{\alpha} \right] \\
&+ \theta_{7,t} \left[a_t n_t^{\alpha} / s_t - \widetilde{c}_t - c_t \right] + \theta_{8,t} \left[\tilde{Z}_t (\varepsilon - 1) / \varepsilon - Z_{1,t} / Z_{2,t} \right] \\
&+ \theta_{9,t} \left[(1 - \phi)(\tilde{Z}_t)^{1-\varepsilon} + \phi \pi_t^{\varepsilon - 1} - 1 \right] + \theta_{10,t} \left[s_t - (1 - \phi) \tilde{Z}_t^{-\varepsilon} - \phi s_{t-1} \pi_t^{\varepsilon} \right] \\
&+ \theta_{11,t} \left[Z_{1,t} - \gamma \widetilde{c}_t^{-\sigma} (a_t n_t^{\alpha} / s_t) mc_t - \phi \beta E_t \pi_{t+1}^{\varepsilon} Z_{1,t+1} \right] \\
&+ \theta_{12,t} \left[Z_{2,t} - \gamma \widetilde{c}_t^{-\sigma} (a_t n_t^{\alpha} / s_t) - \phi \beta E_t \pi_{t+1}^{\varepsilon - 1} Z_{2,t+1} \right] \\
&+ \theta_{13,t} \left[b_t^T - \Gamma b_{t-1}^T / \pi_t \right] + \theta_{14,t} [\kappa_t^B b_{t-1} / (R_t^m \pi_t) - (1 + \Omega_t) m_t^H + m_{t-1}^H \pi_t^{-1}] \Big\},
\end{aligned}$$

given $\tau^n = 1 - (\varepsilon - 1)/(\varepsilon \mu)$ and initial values $b_{-1} > 0$, $b_{-1}^T > 0$, $m_{-1}^H > 0$, and $s_{-1} = 1$.

Proof of Proposition 4. To establish the claims made in the Proposition, we examine first order conditions of (C.1) and show sequentially that several multiplier $\theta_{i,t}$ are equal to zero, indicating that the particular constraints are not binding. The first order condition for κ_t^B is given by $\theta_{14,t}b_{t-1}/\left(R_t^m\pi_t\right) = 0\ \forall t \geq 0$ and implies $\theta_{14,t} = 0$. The first order condition for b_t, $\theta_{5,t} + \beta E_t\theta_{14,t+1}\kappa_{t+1}^B/R_{t+1}^m\pi_{t+1} = 0\ \forall t \geq 0$, then implies $\theta_{5,t} = 0$, and the first order condition for m_t^H, $\theta_{5,t} + (1+\Omega_t)\theta_{4,t} - (1+\Omega_t)\theta_{14,t} + \beta E_t\theta_{14,t+1}/\pi_{t+1} = 0\ \forall t \geq 0$, implies $\theta_{4,t} = 0$. The constraint (A.12) is then irrelevant for the policy problem (and the first order condition for b_t^T, $\theta_{5,t} = \theta_{13,t} - \Gamma\beta E_t\theta_{13,t+1}/\pi_{t+1}$, is consistent with $\theta_{13,t} = 0$). For $R_t^m < c_t^{-\sigma}/(\gamma\widetilde{c}_t^{-\sigma}) \Leftrightarrow$

$$R_t^m < c_t^{-\sigma}/(\beta E_t\left[c_{t+1}^{-\sigma}/\pi_{t+1}\right]), \tag{C.2}$$

the first order condition for κ_t, which requires $\kappa_t > 0$ and reads $\theta_{4,t}l_t/R_t^m = \theta_{3,t}\left(c_t^{\sigma}\gamma\widetilde{c}_t^{-\sigma} - 1/R_t^m\right)$ $\forall t \geq 0$, together with $\theta_{4,t} = 0$ implies $\theta_{3,t} = 0$. Suppose that $\kappa_t > 0$ and (C.2) are satisfied. Then, the first order condition for R_t^L, $\theta_{1,t}mc_ta_t\alpha n_t^{\alpha-1}\left(1/R_t^L\right)^2 - \theta_{3,t}\left(1/R_t^L\right)^2 = 0\ \forall t \geq 0$, together with $\theta_{3,t} = 0$ implies $\theta_{1,t} = 0$. Further, using that the first order condition for l_t, $\theta_{4,t}\kappa_t/R_t^m + \theta_{6,t} = 0$ $\forall t \geq 0$, together with $\theta_{4,t} = 0$ implies $\theta_{6,t} = 0$, we can conclude that the first order condition for mc_t, $-\theta_{1,t}a_t\alpha n_t^{\alpha-1}/R_t^L - \theta_{6,t}a_t\alpha n_t^{\alpha} - \theta_{11,t}\gamma\widetilde{c}_t^{-\sigma}\left(a_tn_t^{\alpha}/s_t\right) = 0\ \forall t \geq 0$, implies $\theta_{11,t} = 0$. Then, the first order conditions for $Z_{1,t}$, $-\theta_{8,t}/Z_{2,0} + \theta_{11,0} = 0$ and $-\theta_{8,t}/Z_{2,t} + \theta_{11,t} - \theta_{11,t-1}\phi\pi_t^{\varepsilon} = 0$ $\forall t \geq 1$, imply $\theta_{8,t} = 0\ \forall t \geq 0$, and the first order conditions for $Z_{2,t}$, $\theta_{8,0}Z_{1,0}/Z_{2,0}^2 + \theta_{12,0} = 0$ and $\theta_{8,t}Z_{1,t}/Z_{2,t}^2 + \theta_{12,t} - \theta_{12,t-1}\phi\pi_t^{\varepsilon-1} = 0\forall t \geq 1$, imply $\theta_{12,t} = 0\ \forall t \geq 0$.

To rewrite the policy problem given in (C.1), we can therefore use that the multiplier $\theta_{4,t}$, $\theta_{5,t}$, $\theta_{13,t}$, and $\theta_{14,t}$ equal zero. We further combine (A.11) and (A.14) to $\frac{1-\phi\pi_t^{\varepsilon-1}}{s_t-\phi s_{t-1}\pi_t^{\varepsilon}} = \tilde{Z}_t$, and substitute out $\tilde{Z}_t$ in (A.14), to get

$$s_t - \phi s_{t-1}\pi_t^{\varepsilon} = (1-\phi)^{\frac{1}{1-\varepsilon}}\left(1-\phi\pi_t^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}}. \tag{C.3}$$

If $\kappa_t > 0$ and (C.2) are satisfied, the multipliers $\theta_{1,t}$, $\theta_{3,t}$ $\theta_{8,t}$, $\theta_{11,t}$, and $\theta_{12,t}$ also equal zero and the remaining constraints for the policy problem are (A.3), (A.13), and (C.3). The policy problem (C.1) can then be rewritten as

$$\begin{aligned}
&\max_{\{c_t,\widetilde{c}_t,n_t,\pi_t,s_t\}_{t=0}^{\infty}}\ \min_{\{\lambda_{1,t},\lambda_{2,t},\lambda_{3,t}\}_{t=0}^{\infty}} \\
&E\sum_{t=0}^{\infty}\beta^t\left\{\left[\left(c_t^{1-\sigma}-1\right)(1-\sigma)^{-1} + \gamma\left(\widetilde{c}_t^{1-\sigma}-1\right)(1-\sigma)^{-1} - \chi n_t^{1+\eta}(1+\eta)^{-1}\right]\right. \\
&+\lambda_{1,t}\left[\beta E_t\left[c_{t+1}^{-\sigma}/\pi_{t+1}\right] - \gamma\widetilde{c}_t^{-\sigma}\right] \qquad + \lambda_{2,t}\left[a_tn_t^{\alpha}/s_t - \widetilde{c}_t - c_t\right] \\
&\left.+\lambda_{3,t}\left[s_t - \phi s_{t-1}\pi_t^{\varepsilon} - (1-\phi)^{\frac{1}{1-\varepsilon}}\left(1-\phi\pi_t^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]\right\}.
\end{aligned} \tag{C.4}$$

As can be seen from (C.4), the allocation under the solution to (C.4) is independent of mark-up shocks μ_t. If (C.2) is not satisfied, i.e. if $R_t^m = c_t^{-\sigma}/(\beta E_t\left[c_{t+1}^{-\sigma}/\pi_{t+1}\right])$ holds, condition (A.2) reduces to $R_t^L = c_t^{-\sigma}/(\beta E_t\left[c_{t+1}^{-\sigma}/\pi_{t+1}\right])$ or $\gamma\widetilde{c}_t^{-\sigma}R_t^L = c_t^{-\sigma}$, such that the multiplier $\theta_{3,t}$ as well as the multipliers $\theta_{1,t}$, $\theta_{8,t}$, $\theta_{11,t}$, and $\theta_{12,t}$ are in general not equal to zero. ■

Consider the policy problem summarized in (C.4). The first order conditions are given by

$$c_t^{-\sigma} = \lambda_{2,t} + \lambda_{1,t-1}\sigma\left(c_t^{-\sigma-1}/\pi_t\right)\ \forall t\geq 1, \quad c_0^{-\sigma} = \lambda_{2,0}, \tag{C.5}$$

$$\gamma\widetilde{c}_t^{-\sigma} = \lambda_{2,t} - \lambda_{1,t}\gamma\sigma\widetilde{c}_t^{-\sigma-1}\ \forall t\geq 0, \tag{C.6}$$

$$\chi n_t^{\eta} = \lambda_{2,t}a_t\alpha n_t^{\alpha-1}/s_t\ \forall t\geq 0, \tag{C.7}$$

$$-\lambda_{1,t-1}c_t^{-\sigma} = \lambda_{3,t}\phi\varepsilon\pi_t^{\varepsilon}\left(s_{t-1}\pi_t - \left[\left(1-\phi\pi_t^{\varepsilon-1}\right)/\left(1-\phi\right)\right]^{\frac{1}{\varepsilon-1}}\right)\ \forall t\geq 1, \tag{C.8}$$

$$0 = \lambda_{3,0}\phi\varepsilon\pi_0^{\varepsilon}\left(s_{-1}\pi_0 - \left[\left(1-\phi\pi_0^{\varepsilon-1}\right)/\left(1-\phi\right)\right]^{\frac{1}{\varepsilon-1}}\right),$$

$$\lambda_{2,t}a_t n_t^{\alpha}/s_t^2 = \lambda_{3,t} - \beta E_t\lambda_{3,t+1}\phi\pi_{t+1}^{\varepsilon}\ \forall t\geq 0, \tag{C.9}$$

as well as (A.3), (A.13), and (C.3). For the subsequent analysis, we abstract from the issue of time-inconsistency which arise here due to the existence of forward-looking private sector equilibrium conditions (see C.5 and C.8). Hence, we disregard choices that would apply only for period $t=0$. Substituting out $\lambda_{2,t}$ with (C.7), the solution to the policy problem (C.4) is then a set of sequences for $\{c_t,\ \widetilde{c}_t,\ n_t,\ \pi_t,\ s_t,\ \lambda_{1,t},\ \lambda_{3,t}\}_{t=0}^{\infty}$ satisfying (A.3), (A.13), (C.3),

$$c_t^{-\sigma}\left[1-\lambda_{1,t-1}\sigma\left(c_t^{-1}/\pi_t\right)\right] = \left(\chi/\alpha\right)n_t^{1-\alpha+\eta}s_t/a_t, \tag{C.10}$$

$$\gamma\widetilde{c}_t^{-\sigma}\left(1+\lambda_{1,t}\sigma\widetilde{c}_t^{-1}\right) = \left(\chi/\alpha\right)n_t^{1-\alpha+\eta}s_t/a_t, \tag{C.11}$$

$$\lambda_{3,t} - \beta\phi E_t\lambda_{3,t+1}\pi_{t+1}^{\varepsilon} = \left(\chi/\alpha\right)n_t^{1+\eta}/s_t, \tag{C.12}$$

$$-\lambda_{1,t-1}c_t^{-\sigma} = \lambda_{3,t}\phi\varepsilon\pi_t^{\varepsilon}\left(s_{t-1}\pi_t - \left[\left(1-\phi\pi_t^{\varepsilon-1}\right)/\left(1-\phi\right)\right]^{\frac{1}{\varepsilon-1}}\right), \tag{C.13}$$

given $s_{-1} = 1$. The <u>steady state</u> under the optimal policy, where exogenous and endogenous variables satisfy $x_t = x_{t-1} = x_{t+1} = x$, is given by a set $\{c,\ \widetilde{c},\ n,\ \pi,\ s,\ \lambda_1,\ \lambda_3\}$ satisfying

$$\gamma\widetilde{c}^{-\sigma} = c^{-\sigma}\left(\beta/\pi\right), \tag{C.14}$$

$$n^{\alpha}/s = \widetilde{c} + c, \tag{C.15}$$

$$s\left(1-\phi\pi^{\varepsilon}\right) = \left(1-\phi\right)^{\frac{1}{1-\varepsilon}}\left(1-\phi\pi^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{C.16}$$

$$c^{-\sigma}\left[1-\lambda_1\sigma\left(c^{-1}/\pi\right)\right] = (\chi/\alpha)n^{1-\alpha+\eta}s, \tag{C.17}$$

$$\gamma\widetilde{c}^{-\sigma}\left(1+\lambda_1\sigma\widetilde{c}^{-1}\right) = c^{-\sigma}\left[1-\lambda_1\sigma\left(c^{-1}/\pi\right)\right], \tag{C.18}$$

$$\lambda_3\left(1-\beta\phi\pi^{\varepsilon}\right) = \left(\chi/\alpha\right)n^{1+\eta}/s, \tag{C.19}$$

$$-\lambda_1 c^{-\sigma}/(\lambda_3\phi\varepsilon\pi^{\varepsilon}) = s\pi - \left[\left(1-\phi\pi^{\varepsilon-1}\right)/\left(1-\phi\right)\right]^{\frac{1}{\varepsilon-1}}. \tag{C.20}$$

Proof of Proposition 5. Suppose that $\kappa_t > 0$ and (C.2) are satisfied, such that the policy problem (C.4) applies and the steady state under the optimal policy is characterized by (C.14)-(C.20). Using (C.14), condition (C.18) can be rewritten as $1-(\beta/\pi) = \lambda_1\sigma\left(c^{-1}/\pi\right)+(\beta/\pi)\,\lambda_1\sigma\widetilde{c}^{-1}$ and further as $\lambda_1 = \frac{1-\beta/\pi}{1+\left(\beta^{1+\sigma}/\gamma\pi\right)^{1/\sigma}}\frac{c\pi}{\sigma}$. Eliminating λ_1 with the latter and λ_3 with (C.19) in (C.20), leads to

$$-\frac{1-\beta/\pi}{1+\left(\beta^{1+\sigma}/\gamma\pi\right)^{1/\sigma}}\frac{\pi}{\sigma}\left\{\frac{c^{1-\sigma}}{(\chi/\alpha)\,n^{1+\eta}}\right\} = \frac{1}{1-\beta\phi\pi^{\varepsilon}}\phi\varepsilon\pi^{\varepsilon}\left[\pi-\frac{1}{s}\left(\frac{1-\phi\pi^{\varepsilon-1}}{1-\phi}\right)^{\frac{1}{\varepsilon-1}}\right]$$
$$=\frac{\phi\varepsilon\pi^{\varepsilon}}{1-\beta\phi\pi^{\varepsilon}}\frac{\pi-1}{1-\phi\pi^{\varepsilon-1}}, \qquad \text{(C.21)}$$

where we eliminated s with (C.16) for the second equality. Substituting out λ_1 as above and s with (C.15) in (C.17) gives $c^{-\sigma}\frac{\beta/\pi+\left(\beta^{1+\sigma}/\gamma\pi\right)^{1/\sigma}}{1+\left(\beta^{1+\sigma}/\gamma\pi\right)^{1/\sigma}} = (\chi/\alpha)\,n^{\,\eta+1}\frac{1}{\widetilde{c}+c}$, which after eliminating $\widetilde{c}$ with (C.14) reads

$$\frac{c^{1-\sigma}}{(\chi/\alpha)\,n^{\,\eta+1}} = \frac{1}{1+(\gamma\pi/\beta)^{1/\sigma}}\frac{1+\left(\beta^{1+\sigma}/\gamma\pi\right)^{1/\sigma}}{(\beta/\pi)+\left(\beta^{1+\sigma}/\gamma\pi\right)^{1/\sigma}}. \qquad \text{(C.22)}$$

Substituting out the term in the curly brackets on the LHS of (C.21) with (C.22) gives the following condition, which features the steady state inflation rate as the single unknown:

$$\frac{1-\pi}{(1-\phi\pi^{\varepsilon-1})\,(1-\beta\phi\pi^{\varepsilon})}\sigma\varepsilon\phi = \Theta(\pi)\,(1-\beta/\pi)\,, \qquad \text{(C.23)}$$
$$\text{where } \Theta(\pi) = \frac{(\gamma\pi)^{1/\sigma}}{\left[(\gamma\pi)^{1/\sigma}\,\beta/\pi\right]+\beta^{1+1/\sigma}}\frac{\pi^{1-\varepsilon}}{1+(\gamma\pi/\beta)^{1/\sigma}}.$$

For $\gamma > 0$, $\Theta(\pi) > 0$ if $\pi \in (0,\infty)$. Suppose that $\pi = 1$. Then, the LHS of (C.23) would be equal to zero, given that $\beta \in (0,1)$ and $\phi \in (0,1)$. For the RHS of (C.23) also to equal zero, $\Theta(\pi) > 0$ would demand $\pi = \beta$, which is a contradiction, since $\beta \in (0,1)$. Neither $\pi = 1$ nor $\pi = \beta$ therefore solve (C.23), such that prices are not stable in the long-run under the optimal policy and the allocation differs from the first best allocation (29), since $\gamma\widetilde{c}^{-\sigma} \neq c^{-\sigma}$ according to (C.14). ■

Proof of Proposition 6. Suppose that $\kappa_t > 0$ and (C.2) are satisfied, such that the policy problem (C.4) applies, that there are no credit goods, $\gamma = 0$, and that the production subsidy τ^n satisfies (37). Then, the equilibrium condition (A.3), which is the first constraint in (C.4) associated with the multiplier $\lambda_{1,t}$, becomes irrelevant and the set of conditions (A.3), (A.13), (C.3), (C.10)-(C.13) describing the solution to (C.4) simplifies to the following set of conditions for c_t, n_t, π_t, s_t, and $\lambda_{3,t}$:

$$a_t n_t^{\alpha}/s_t = c_t, \tag{C.24}$$

$$c_t^{-\sigma} = (\chi/\alpha) n_t^{1-\alpha+\eta} s_t/a_t, \tag{C.25}$$

$$0 = \lambda_{3,t}\phi\varepsilon\pi_t^{\varepsilon}\left(s_{t-1}\pi_t - \left[\left(1-\phi\pi_t^{\varepsilon-1}\right)/\left(1-\phi\right)\right]^{\frac{1}{\varepsilon-1}}\right), \tag{C.26}$$

(C.3), and (C.12), which hold $\forall t \geq 0$. For $\lambda_{3,t} = 0$, the constraint (C.3) is not binding and the policy problem (C.4) reduces to the one of a social planer, leading to the first best allocation. For $\lambda_{3,t} \neq 0$, condition (C.26) implies $s_{t-1} = \pi_t^{-1}\left[\left(1-\phi\pi_t^{\varepsilon-1}\right)/\left(1-\phi\right)\right]^{\frac{1}{\varepsilon-1}}$. Substituting out s_{t-1} with the latter in (C.3) gives

$$s_t^{\varepsilon-1} = \left(1-\phi\pi_t^{\varepsilon-1}\right)/\left(1-\phi\right), \tag{C.27}$$

which can be combined with (C.3) to get $s_t - \phi s_{t-1}\pi_t^{\varepsilon} = s_t\left(1-\phi\pi_t^{\varepsilon-1}\right)$, implying $s_t/s_{t-1} = \pi_t$. Eliminating π_t with the latter in (C.27) leads to $(s_t/s_{t-1})^{\varepsilon-1} = \left(1-(1-\phi)s_t^{\varepsilon-1}\right)/\phi \Leftrightarrow s_t^{1-\varepsilon} = \phi s_{t-1}^{1-\varepsilon} + (1-\phi)$, which by iterating backwards implies $x_t = \phi^{t+1}x_{-1} + \left(1-\phi^{t+1}\right)$, where $x_t = s_t^{1-\varepsilon}$ and we used $(1-\phi)\sum_{i=0}^{t}\phi^i = \left(1-\phi^{t+1}\right)$. Given that $s_{-1} = 1$ and thus $x_{-1} = 1$, we can conclude that $s_t = 1$ $\forall t \geq 0$, and – by $s_t/s_{t-1} = \pi_t$ – that $\pi_t = 1$. Then, (C.24) and (C.25) reduce to $a_t n_t^{\alpha} = c_t$ and $\chi n_t^{\eta}/c_t^{-\sigma} = a_t\alpha n_t^{\alpha-1}$, which are identical to the conditions describing the first best allocation for $\gamma = 0$ (see 29). ■

Proof of Proposition 7. As shown in the proof of Proposition 6, the allocation and prices under the optimal policy for $\gamma = 0$ and (37) satisfy $n_t^{\alpha}/s_t = c_t$, $\chi n_t^{1+\eta-\alpha} = a_t\alpha c_t^{-\sigma}$, $s_t = 1$, and $\pi_t = 1$. To show how this can be implemented by the central bank, we substitute out $\tilde{Z}_t$ by (A.10) in (A.11), where $Z_{1,t} = \lambda_t y_t mc_t + \phi\beta E_t\pi_{t+1}^{\varepsilon}Z_{1,t+1}$, $Z_{2,t} = \lambda_t y_t + \phi\beta E_t\pi_{t+1}^{\varepsilon-1}Z_{2,t+1}$, and $\lambda_t = \beta E_t\frac{c_t^{-\sigma}}{\pi_{t+1}}$ (see 12, 18, and 19), to get

$$\left[\left(1-\phi\pi_t^{\varepsilon-1}\right)/\left(1-\phi\right)\right]^{1/(1-\varepsilon)} = \frac{\varepsilon}{\varepsilon-1}\frac{E_t\sum_{s=0}^{\infty}(\phi\beta)^s\left(\Pi_{k=1}^{s}\pi_{t+k}^{\varepsilon}\right)\lambda_{t+s}y_{t+s}mc_{t+s}}{E_t\sum_{s=0}^{\infty}(\phi\beta)^s\left(\Pi_{k=1}^{s}\pi_{t+k}^{\varepsilon-1}\right)\lambda_{t+s}y_{t+s}}, \tag{C.28}$$

for which we iterated $Z_{1,t} = \lambda_t y_t mc_t + \phi\beta E_t\pi_{t+1}^{\varepsilon}Z_{1,t+1}$ and $Z_{2,t} = \lambda_t y_t + \phi\beta E_t\pi_{t+1}^{\varepsilon-1}Z_{2,t+1}$ forward. For $mc_{t+s} = \frac{\varepsilon-1}{\varepsilon}$, price stability $\pi_t = 1$ solves (C.28), which together with (C.3) and $s_{-1} = 1$ implies $s_t = 1$. To implement this solution, we use that the constraints (A.4), (A.5), and (A.6) are binding under (C.2) (see 26 and 28). First, the instruments $\kappa_t > 0$ and R_t^m are set to control the loan rate according to (A.2) in a way that it satisfies $R_t^L/R^L = \mu/\mu_t$,

$$(1-\kappa_t)\beta c_t^{\sigma}E_t[c_{t+1}^{-\sigma}\pi_{t+1}^{-1}] + \kappa_t/R_t^m = \mu_t/\left(\mu R^L\right). \tag{C.29}$$

Then, (38) reduces to $\chi n_t^{\eta}/c_t^{-\sigma} = a_t\alpha n_t^{\alpha-1}[mc_t/mc]$. Second, the instruments κ_t^B, κ_t, and R_t^m are set to implement particular values for aggregate demand $\overline{c}_t$ and thus working time $\overline{n}_t$, which imply $mc_t = \frac{\varepsilon-1}{\varepsilon}$. These values can be identified by solving for consumption and working time using (C.24) and (C.25): $\overline{c}_t = s_t^{-(1+\eta)/(\eta+\alpha\sigma+1-\alpha)} c_t^*$ and $\overline{n}_t = s_t^{(\sigma-1)/(\eta+\alpha\sigma+1-\alpha)} n_t^*$, where c_t^* and n_t^* denote the first best values (see 29). Substituting out working time and the subsidy τ^n in (34) with (C.24) and (37), then shows that the central bank can set its instruments κ_t^B, κ_t, and R_t^m to ration money supply and thus consumption up to the level $\overline{c}_t$ by satisfying

$$\kappa_t^B b_{t-1} = R_t^m \pi_t \left[\overline{c}_t \left(1 - \kappa_t \frac{mc_t \alpha s_t \mu R^L}{mc R_t^m} \right) - m_{t-1}^H \pi_t^{-1} \right]. \tag{C.30}$$

Inserting $\overline{c}_t$ and $\overline{n}_t$ into $\chi n_t^\eta / c_t^{-\sigma} = a_t \alpha n_t^{\alpha-1} [mc_t/mc]$, then leads to $mc_t = mc$. Given that the latter implies $\pi_t = 1$ and $s_t = 1$, the values $\overline{c}_t$ and $\overline{n}_t$ will be identical to the first best values $\overline{c}_t = c_t^*$ and $\overline{n}_t = n_t^*$ in equilibrium. For $\kappa_t = 0$, (C.29) reduces to $\beta c_t^\sigma E_t [c_{t+1}^{-\sigma} \pi_{t+1}^{-1}] = \mu_t / \left(\mu R^L\right)$, which cannot be satisfied under first best, since $\pi_t = 1$ would then hold and c_t^* is not a function of μ_t (see 29).

To ensure price stability in the long-run ($t \to \infty$), we consider the deterministic steady state under the optimal policy, where $a_t = 1$ and $\mu_t = \mu$ holds and the allocation is time-invariant, $n = (\alpha/\chi)^{\frac{1}{\eta+1-\alpha+\sigma\alpha}}$ and $c = n^\alpha$ (see 29). Suppose that the instruments R_t^m and κ_t are constant in the long-run, whereas κ_t^B and Ω_t are adjusted in the long-run to implement price stability. Then, (A.2) and (B.2) imply that R_t^L and mc_t are also constant in the long-run. Since (C.2) holds under the optimal policy, (A.4)-(A.6) are binding. Then, long-run money holdings satisfy $(1+\Omega_t) m_t^H = c - \kappa \left(\alpha n^\alpha \mu R^L / R^m\right)$, where we used (A.8), (A.9), and (37), and (B.1) implies $\kappa_t^B b_{t-1} = R^m \left[c - \kappa \left(\alpha n^\alpha \mu R^L / R^m\right)\right] \left(\pi - (1+\Omega_{t-1})^{-1}\right)$. Substituting out household bond and money holdings in (A.7) then gives $b_{t-1}^T = \left[c - \kappa \left(\alpha n^\alpha \mu R^L / R^m\right)\right] (1 + \Omega_{t-1})^{-1} \left[\left(R^m/\kappa_t^B\right) \left(\pi \left(1+\Omega_{t-1}\right) - 1\right) + 1\right]$, which is used to eliminate total bonds in (A.12), implying for $t \to \infty$:

$$\left(\frac{R^m}{\kappa_{t+1}^B} \left[\pi \left(1+\Omega_t\right) - 1\right] + 1 \right) = \Gamma \pi^{-1} \frac{1+\Omega_t}{1+\Omega_{t-1}} \left(\frac{R^m}{\kappa_t^B} \left[\pi \left(1+\Omega_{t-1}\right) - 1\right] + 1 \right). \tag{C.31}$$

Hence, for the implementation of long-run price stability, $\pi = 1$, (C.31) demands that the instruments $\kappa_t^B \in [0,1)$ and $\Omega_t \geq 0$ have to satisfy

$$\lim_{t\to\infty} \left(\frac{1 + R^m \Omega_t / \kappa_{t+1}^B}{1+\Omega_t} - \Gamma \frac{1 + R^m \Omega_{t-1} / \kappa_t^B}{1+\Omega_{t-1}} \right) = 0, \tag{C.32}$$

For $\Gamma > 1$, (C.32) can be satisfied by $\Omega_t = \Omega > 0$ and by letting κ_t^B shrink according to $\lim_{t\to\infty} \frac{1}{\kappa_{t+1}^B} - \Gamma \frac{1}{\kappa_t^B} - \frac{\Gamma-1}{R^m \Omega} = 0$, while for $\Gamma < 1$ and $\kappa_t^B = \kappa^B \in (0,1]$ by letting Ω_t grow according to $\lim_{t\to\infty} \frac{1}{1+\Omega_t} - \Gamma \frac{1}{1+\Omega_{t-1}} - \frac{1-\Gamma}{1-\kappa^B/R^m} = 0$. For $\Gamma = 1$, constant values for Ω_t and κ_t^B satisfy (C.32). ■

APPENDICES NOT INTENDED FOR PUBLICATION

D Retained Earnings

In this Appendix, we examine if intermediate goods producing firms choose to borrow from households, such that (1) holds, even when retained earnings are considered. When a firm j exists for more than one period, it can retain earnings to the amount $H_{j,t}$ at the end of each period, which can be used to finance the wage bill in the next period. Then, its nominal profits $P_t v^f_{j,t}$ are

$$P_t v^f_{j,t} = (1+\tau^p) P_{J,t} a_t n^\alpha_{j,t} - P_t w_t n_{j,t} + L_{j,t} \left(1/R^L_t - 1\right) - H_{j,t} + H_{j,t-1}. \tag{D.1}$$

Instead of (1), its working capital constraint would then be given by

$$\left(L_{j,t}/R^L_t\right) + H_{j,t-1} \geq P_t w_t n_{j,t}. \tag{D.2}$$

Firm j maximizes the present value of profits subject to (D.1), (D.2), and a non-negativity constraint on retained earnings $H_{j,t} \geq 0$, such that its problem can be summarized as

$$\max_{\{n_{j,t}, H_{j,t}, L_{j,t}\}^\infty_{t=0}} E_t \sum_{k=0}^{\infty} q_{t,t+k} v^f_{j,t+s} \text{ s.t. (D.1), (D.2), and } H_{j,t+s} \geq 0.$$

The solution to this problem is $\forall k \geq 0$ characterized by $P_{J,t+k} a_{t+k} \alpha n^{\alpha-1}_{j,t+k} = (1+\varsigma_{j,t+k}) P_{t+k} w_{t+k}$, $\varpi_{j,t+k} = 1 - \beta E_{t+k} \frac{q_{t,t+k+1}}{q_{t,t+k}} (1+\varsigma_{j,t+k+1})$, and $\varsigma_{j,t+k} = R^L_{t+k} - 1$, as well as by the complementary slackness conditions $\varsigma_{j,t+k}[(L_{j,t+k}/R^L_{t+k}) + H_{j,t+k-1} - P_{t+k} w_{t+k} n_{j,t+k}] = 0$, $\varsigma_{j,t+k} \geq 0$, $\varpi_{j,t+k} h_{j,t+k} \geq 0$, and $\varpi_{j,t+k} \geq 0$. Eliminating $\varsigma_{j,t+k}$ then leads to (6) and

$$\varpi_{j,t+k} = 1 - \beta E_{t+k} \left[(q_{t,t+k+1}/q_{t,t+k}) R^L_{t+k+1}\right],$$

which shows that firms are not willing to retain earnings, $\varpi_{j,t+k} > 0 \Rightarrow H_{j,t+k} = 0$, if the loan rate is sufficiently low, $q^{-1}_{t,t+k} \beta E_{t+k} \left[q_{t,t+k+1} R^L_{t+k+1}\right] < 1$. In equilibrium, the latter implies $\widetilde{c}^\sigma_t \beta E_t \left[\widetilde{c}^{-\sigma}_{t+1} R^L_{t+1}/\pi_{t+1}\right] < 1$ and for the steady state

$$R^L < \pi/\beta. \tag{D.3}$$

As can be seen from (22), (D.3) is satisfied if the central bank instruments are set according to $\kappa > 0$ and $R^m < \pi/\beta$ in the long-run. These restrictions on the policy instruments are in fact satisfied under optimal policy, as shown in Proposition 4. Hence, as long as deviations from this steady state are sufficiently small, which is implicitly assumed for the shocks processes, $\varpi_{j,t} > 0$ holds under the optimal policy. Then, loans satisfy (7), which holds with equality if $R^L_{t+k} > 1$.

E Optimal Policy without Money Rationing

In this Appendix, we derive the optimal policy plan under commitment for the conventional case where money supply is not rationed. For this, we refer to the policy problem (C.1) and consider the case where (C.2) is not satisfied, such that $R_t^m = c_t^{-\sigma}/(\beta E_t\left[c_{t+1}^{-\sigma}/\pi_{t+1}\right])$. Then, the collateral constraint (27) is not binding (see 28), and the multipliers $\theta_{1,t}$, $\theta_{3,t}$ $\theta_{8,t}$, $\theta_{11,t}$, and $\theta_{12,t}$ in (C.1) are in general not equal to zero (see proof of Proposition 4). The policy problem (C.1) is then given by

$$\max_{\{c_t,\widetilde{c}_t,n_t,mc_t,\tilde{Z}_t,Z_{1,t},Z_{2,t},s_t,\pi_t\}_{t=0}^{\infty}} \min_{\{\lambda_{0,t},\ldots\lambda_{7,t}\}_{t=0}^{\infty}} \tag{E.1}$$

$$\begin{aligned}
\mathcal{L}_t = E\sum_{t=0}^{\infty}\beta^t \Big\{ & \left[\left(c_t^{1-\sigma}-1\right)(1-\sigma)^{-1} + \gamma(\widetilde{c}_t^{1-\sigma}-1)(1-\sigma)^{-1} - \chi n_t^{1+\eta}(1+\eta)^{-1}\right] \\
& +\lambda_{0,t}\left[\left[\mu_t(1-\tau^n)\chi n_t^{\eta+1-\alpha}/(mc_t\alpha a_t)\right] - \beta E_t\left(c_{t+1}^{-\sigma}/\pi_{t+1}\right)\right] \\
& +\lambda_{1,t}\left[\beta E_t\left(c_{t+1}^{-\sigma}/\pi_{t+1}\right) - \gamma\widetilde{c}_t^{-\sigma}\right] + \lambda_{2,t}\left[(a_t n_t^{\alpha}/s_t) - \widetilde{c}_t - c_t\right] \\
& +\lambda_{3,t}\left[s_t - \phi s_{t-1}\pi_t^{\varepsilon} - (1-\phi)^{\frac{1}{1-\varepsilon}}\left(1-\phi\pi_t^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right] \\
& +\lambda_{4,t}\left[(1-\phi)(\tilde{Z}_t)^{1-\varepsilon} + \phi\pi_t^{\varepsilon-1} - 1\right] + \lambda_{5,t}\left[\tilde{Z}_t\left\{(\varepsilon-1)/\varepsilon\right\} - Z_{1,t}/Z_{2,t}\right] \\
& +\lambda_{6,t}\left[Z_{1,t} - \gamma\widetilde{c}_t^{-\sigma}\left(a_t n_t^{\alpha}/s_t\right)mc_t - \phi\beta E_t\pi_{t+1}^{\varepsilon}Z_{1,t+1}\right] \\
& +\lambda_{7,t}\left[Z_{2,t} - \gamma\widetilde{c}_t^{-\sigma}\left(a_t n_t^{\alpha}/s_t\right) - \phi\beta E_t\pi_{t+1}^{\varepsilon-1}Z_{2,t+1}\right]\Big\}.
\end{aligned}$$

Neglecting the conditions for period $t=0$, the solution to the policy problem (E.1) has to satisfy the following first order conditions

$$\begin{aligned}
0 &= c_t^{-\sigma}\left(1+\sigma c_t^{-1}\left(\lambda_{0,t-1}-\lambda_{1,t-1}\right)/\pi_t\right) - \lambda_{2,t}, \\
0 &= \gamma\widetilde{c}_t^{-\sigma} + \lambda_{1,t}\gamma\sigma\widetilde{c}_t^{-\sigma-1} - \lambda_{2,t} + \lambda_{6,t}\gamma\sigma\widetilde{c}_t^{-\sigma-1}\left(a_t n_t^{\alpha}/s_t\right)mc_t + \lambda_{7,t}\gamma\sigma\widetilde{c}_t^{-\sigma-1}\left(a_t n_t^{\alpha}/s_t\right), \\
0 &= -\chi n_t^{\eta} + \left[\lambda_{0,t}\left(\eta+1-\alpha\right)\mu_t(1-\tau^n)\chi n_t^{\eta-\alpha}/\left(mc_t\alpha a_t\right)\right] + \left(\lambda_{2,t}a_t\alpha n_t^{\alpha-1}/s_t\right) \\
&\quad -\lambda_{6,t}\alpha\gamma\widetilde{c}_t^{-\sigma}\left(a_t n_t^{\alpha-1}/s_t\right)mc_t - \lambda_{7,t}\alpha\gamma\widetilde{c}_t^{-\sigma}\left(a_t n_t^{\alpha-1}/s_t\right), \\
0 &= \left(\lambda_{0,t-1}c_t^{-\sigma}/\pi_t^2\right) - \left(\lambda_{1,t-1}c_t^{-\sigma}/\pi_t^2\right) + \lambda_{4,t}\left(\varepsilon-1\right)\phi\pi_t^{\varepsilon-2} - \lambda_{6,t-1}\phi\varepsilon\pi_t^{\varepsilon-1}Z_{1,t} - \lambda_{7,t-1}\phi\left(\varepsilon-1\right)\pi_t^{\varepsilon-2}Z_{2,t} \\
&\quad +\lambda_{3,t}\left[-\phi s_{t-1}\varepsilon\pi_t^{\varepsilon-1} - (1-\phi)^{\frac{1}{1-\varepsilon}}\left(\varepsilon/(\varepsilon-1)\right)\left(1-\phi\pi_t^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}\left(-\left(\varepsilon-1\right)\phi\pi_t^{\varepsilon-2}\right)\right], \\
0 &= -\left(\lambda_{2,t}a_t n_t^{\alpha}/s_t^2\right) + \lambda_{3,t} - \beta E_t\lambda_{3,t+1}\phi\pi_{t+1}^{\varepsilon} + \lambda_{6,t}\gamma\widetilde{c}_t^{-\sigma}\left(a_t n_t^{\alpha}/s_t^2\right)mc_t + \lambda_{7,t}\gamma\widetilde{c}_t^{-\sigma}\left(a_t n_t^{\alpha}/s_t^2\right), \\
0 &= -\left[\lambda_{0,t}\mu_t(1-\tau^n)\chi n_t^{\eta+1-\alpha}/\left(mc_t^2\alpha a_t\right)\right] - \lambda_{6,t}\gamma\widetilde{c}_t^{-\sigma}\left(a_t n_t^{\alpha}/s_t\right), \\
0 &= -\left(\lambda_{5,t}/Z_{2,t}\right) + \lambda_{6,t} - \lambda_{6,t-1}\phi\pi_t^{\varepsilon}, \\
0 &= \left(\lambda_{5,t}Z_{1,t}/Z_{2,t}^2\right) + \lambda_{7,t} - \lambda_{7,t-1}\phi\pi_t^{\varepsilon-1}, \\
0 &= \lambda_{4,t}(1-\phi)\left(1-\varepsilon\right)(\tilde{Z}_t)^{-\varepsilon} + \lambda_{5,t}\left(\varepsilon-1\right)/\varepsilon,
\end{aligned}$$

which together with $\mu_t(1-\tau^n)\chi n_t^{\eta+1-\alpha}/(mc_t\alpha a_t) = \beta E_t\left(c_{t+1}^{-\sigma}/\pi_{t+1}\right)$, (A.3), (A.10) including the definitions for $Z_{1,t}$ and $Z_{2,t}$, (A.11), (A.13), and (A.14) describe the solution for $\{\lambda_{0\,t},\ \lambda_{1\,t},\ \lambda_{2,t},\ \lambda_{3\,t},\ \lambda_{4\,t},\ \lambda_{5\,t},\ \lambda_{6\,t},\ \lambda_{7\,t},\ c_t,\ \widetilde{c}_t,\ n_t,\ mc_t,\ \tilde{Z}_t,\ Z_{1,t},\ Z_{2,t},\ s_t,\ \pi_t\}_{t=0}^{\infty}$ given $s_{-1}=1$. Eliminating $\lambda_{2,t}$, $\lambda_{4\,t}$, $\lambda_{5\,t}$, $\lambda_{6\,t}$, $\lambda_{7\,t}$, mc_t, $\tilde{Z}_t$, $Z_{1,t}$, and $Z_{2,t}$, the steady state of the solution to (E.1), where all exogenous and endogenous variables satisfy $x_t = x_{t-1} = x_{t+1} = x$, can be reduced to a set $\{c,\ \widetilde{c},\ n,\ \pi,\ s,\ \lambda_0,\ \lambda_1,\ \lambda_3\}$ satisfying (C.14)-(C.16), and

$$
\begin{aligned}
0 &= \beta + \lambda_1\sigma\left(c^{-1}+\beta\widetilde{c}^{-1}\right) - \lambda_0\sigma\left(c^{-1}+\beta\widetilde{c}^{-1}\Theta(\pi)\right) - \pi,\\
0 &= \frac{\pi}{\beta}\left(1+\sigma c^{-1}\left(\lambda_0-\lambda_1\right)/\pi\right)\left(\alpha n^{\alpha}/s\right) + \lambda_0\left(\eta+1-\alpha+\alpha\Theta(\pi)\right) - \frac{\chi n^{\eta+1}}{c^{-\sigma}}\frac{\pi}{\beta},\\
0 &= \lambda_0\Theta(\pi) + \frac{\pi}{\beta}\left(1+\sigma c^{-1}\left(\lambda_0-\lambda_1\right)/\pi\right)\left(n^{\alpha}/s\right) - \frac{\pi}{\beta}s\lambda_3 c^{\sigma}\left(1-\beta\phi\pi^{\varepsilon}\right),\\
0 &= \lambda_1 + \lambda_3\phi\varepsilon\pi^{\varepsilon}\frac{s}{c^{-\sigma}}\frac{\pi-1}{1-\phi\pi^{\varepsilon-1}} - \lambda_0\left(1-\beta\frac{\varepsilon\phi\pi^{\varepsilon-1}}{1-\phi\beta\pi^{\varepsilon}}\frac{1-\pi}{1-\pi^{\varepsilon-1}\phi}\right),\\
0 &= \mu\frac{\varepsilon}{\varepsilon-1}(1-\tau^n)\frac{\pi}{\beta}\left(\frac{1-\phi\pi^{\varepsilon-1}}{1-\phi}\right)^{\frac{1}{\varepsilon-1}}\frac{\left(1-\phi\beta\pi^{\varepsilon-1}\right)}{\left(1-\phi\beta\pi^{\varepsilon}\right)} - \frac{c^{-\sigma}}{\left(\chi/\alpha\right)n^{\eta+1-\alpha}},
\end{aligned}
$$

where $\Theta(\pi)$ is defined as $\Theta(\pi) = 1 - \frac{1-\phi\pi^{\varepsilon}}{1-\phi\pi^{\varepsilon-1}}\frac{1-\phi\beta\pi^{\varepsilon-1}}{1-\phi\beta\pi^{\varepsilon}}$.

F A Version with Consumption Loans

In this Appendix, we examine an alternative version of the model to demonstrate the robustness of the main result, namely, that the central bank can enhance welfare via money rationing. For this, we neglect the working capital constraint (1) and consider borrowing and lending between households. We assume that households differ with regard to their marginal valuation of the cash good due to preference shocks $\epsilon_{i,t}$, while we abstract from consumption of credit goods (which accords to $\gamma = 0$). Specifically, households' instantaneous utility function is now given by

$$
u(c_{i,t}, n_{i,t}, \epsilon_{i,t}) = \epsilon_{i,t}(c_{i,t}^{1-\sigma}-1)\left(1-\sigma\right)^{-1} - \chi n_{i,t}^{1+\eta}\left(1+\eta\right)^{-1}, \tag{F.1}
$$

where $\sigma > 0$, $\chi > 0$, and $\eta \geq 0$. To simplify the analysis, we assume that the stochastic component $\epsilon_{i,t} > 0$ is i.i.d. with mean one, such that households are ex-ante identical in every period (see below). We will restrict our attention to the case where $\epsilon_{i,t}$ exhibits two possible realizations, $\epsilon_{i,t} \in \{\epsilon_b,\ \epsilon_l\}$, with equal probabilities, where $\epsilon_l < \epsilon_b$. These idiosyncratic shocks as well as the aggregate shocks materialize at the beginning of each period and the timing of events in each period is unchanged (see Section 2.1).

Households again face a cash constraint on consumption, which cause them to hold money and assets eligible for open market operations. We further consider a market for intraperiod consumption loans, where households can borrow and lend cash to the amount $L_{i,t} \lesseqqgtr 0$ at the price $1/R_t^L$. When household i draws the realization ϵ_b (ϵ_l), it tends to consume more (less) than households who draw ϵ_l (ϵ_b) and to borrow (lend) cash. Its cash-in-advance constraint is given by

$$P_t c_{i,t} \leq I_{i,t} + M^H_{i,t-1} - \left(L_{i,t}/R^L_t\right), \tag{F.2}$$

(instead of 3), where a negative (positive) value for $L_{i,t}$ indicates borrowing (lending). The amount of cash that can be acquired from the central bank in exchange for treasuries and consumption loans satisfies $I_{i,t} \leq \kappa_t \left(L_{i,t}/R^m_t\right) + \kappa^B_t \left(B_{i,t-1}/R^m_t\right)$ for a lending household with $L_{i,t} \geq 0$ and $I_{i,t} \leq \kappa^B_t B_{i,t-1}/R^m_t$ for a borrowing household with $L_{i,t} < 0$. The budget constraint of a household i is given by (11), where $P_t \widetilde{c}_{i,t}$ equals zero. Maximizing $E \sum_{t=0}^{\infty} \beta^t u(c_{i,t}, n_{i,t}, \epsilon_{i,t})$ subject to (2) for $L_{i,t} \geq 0$ and $I_{i,t} \leq \kappa^B_t B_{i,t-1}/R^m_t$ for $L_{i,t} < 0$, (10), (11) for $\widetilde{c}_{i,t} = 0$, (F.2), and the borrowing constraints $\lim_{s\to\infty} E_t q_{t,t+s} D_{i,t+s} \geq 0$, $M^H_{i,t} \geq 0$ and $B_{i,t} \geq 0$, leads to the first order conditions $\epsilon_{i,t} c^{-\sigma}_{i,t} = \lambda_{i,t} + \psi_{i,t}$, $\mu_t \chi n^{\eta}_{i,t} = w_t \lambda_{i,t}$, as well as (15), (17)-(19), and

$$\psi_{i,t} = \lambda_{i,t}\left(R^L_t - 1\right) + R^L_t \kappa_t \eta_{i,t} \text{ if } L_{i,t} \geq 0 \quad \text{or} \quad \psi_{i,t} = \lambda_{i,t}\left(R^L_t - 1\right) \text{ if } L_{i,t} < 0 \tag{F.3}$$

(where the multipliers $\lambda_{i,t}$, $\psi_{i,t}$, and $\eta_{i,t}$ are defined as in Section 2.3), the associated complementary slackness conditions, and the transversality conditions. Given that wage payments are not liquidity constrained, an inflation tax now distorts the consumption/labor decision, which can be seen, after eliminating the multipliers in (18) with $\epsilon_{i,t} c^{-\sigma}_{i,t} = \lambda_{i,t} + \psi_{i,t}$ and $\mu_t \chi n^{\eta}_{i,t} = w_t \lambda_{i,t}$, from

$$\mu_t \chi n^{\eta}_{i,t} = w_t \beta E_t \left[\epsilon_{i,t+1} c^{-\sigma}_{i,t+1}/\pi_{t+1}\right]. \tag{F.4}$$

Due to the assumption that preference shocks are i.i.d., the RHS of condition (F.4) is identical for all households and labor supplies satisfy, $n_{l,t} = n_{b,t}$. Likewise, money and bond holdings of households are identical, e.g. $M^H_{b,t} = M^H_{l,t}$ and $B_{b,t} = B_{l,t}$. The first order conditions for loans (F.3) can further be transformed to $1/R^L_t = (c^{\sigma}_{b,t}/\epsilon_{b,t})\beta E_t[\epsilon_{i,t+1} c^{-\sigma}_{i,t+1}/\pi_{t+1}]$ and $1/R^L_t = (\kappa_t/R^m_t) + (1-\kappa_t)\,(c^{\sigma}_{l,t}/\epsilon_{l,t})\beta E_t[\epsilon_{i,t+1} c^{-\sigma}_{i,t+1}/\pi_{t+1}]$. Combining these conditions shows that the marginal utilities of consumption of the two household types can differ, $\epsilon_{b,t} c^{-\sigma}_{b,t} < \epsilon_{l,t} c^{-\sigma}_{l,t}$, if the policy rate is lower than the loan rate and loans are eligible with $\kappa_t \in (0,1)$:

$$\epsilon_{b,t} c^{-\sigma}_{b,t} = \frac{1 - \kappa_t (R^L_t/R^m_t)}{1-\kappa_t} \epsilon_{l,t} c^{-\sigma}_{l,t}. \tag{F.5}$$

Compared to the case where loans are not eligible, $\kappa_t = 0$, which implies $\epsilon_{b,t} c^{-\sigma}_{b,t} = \epsilon_{l,t} c^{-\sigma}_{l,t}$ (see F.5),[33] the volume of loans and, therefore, relative consumption of borrowers increase with larger ratios R^L_t/R^m_t for $\kappa_t > 0$ and thus with lower refinancing costs of lenders.

The remaining equilibrium conditions are unchanged, except of the firms' labor demand condi-

[33] Note that the identity of both types' marginal utilities of consumption is then shared with the first best allocation. Nevertheless, $\kappa_t = 0$ will not be chosen under an optimal policy due to the existence of further distortions.

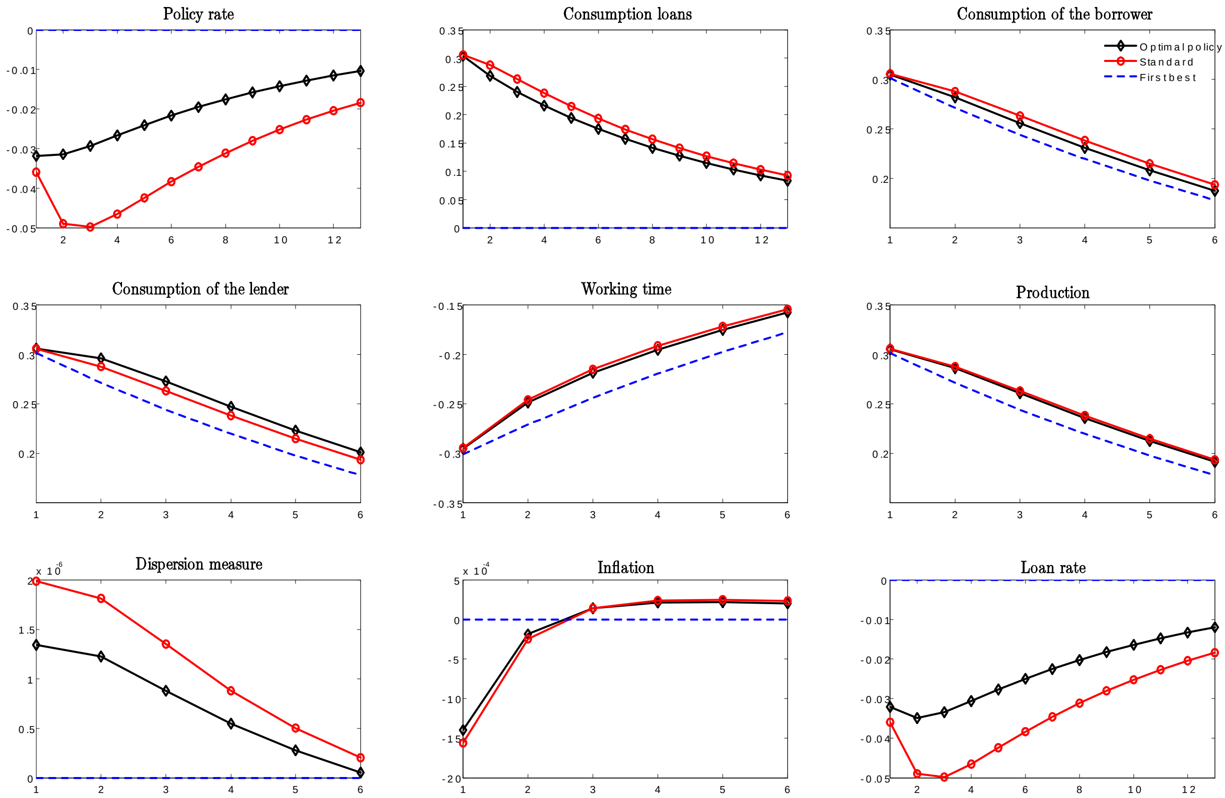

Figure 5: Responses (in % dev. from the steady states) to a productivity shock for the consumption loans model

tion (6), which changes to $(P_{J,t}/P_t)\, a_t \alpha n_{j,t}^{\alpha-1} = (1-\tau^n)\, w_t$, and the firms' loan demand (7), which is irrelevant. The full set of equilibrium conditions and the derivation of optimal policy can be found in Appendix G. For the derivation of the optimal policy and for the else optimal policy under non-rationed money supply, we consider a wage subsidy that compensates for the average price and wage mark-ups, $\tau^n = 1 - \frac{\varepsilon-1}{\varepsilon\mu}$ (like in Section 5). Corresponding to the benchmark model, optimal policy (*opt*) is associated with $\kappa_t > 0$ and $R_t^m < \epsilon_{l,t} c_{l,t}^{-\sigma} / (\beta E_t[\epsilon_{i,t+1} c_{i,t+1}^{-\sigma}/\pi_{t+1}])$, such that (2) is binding and money supply is rationed. This tends to lower the loan rate and thereby the costs of borrowing cash compared to the standard optimal policy (*std*). For the numerical solution of the model, we apply the same parameter values as before. The only difference (beside $\gamma = 0$) to the parameter values described in Section 5 is the value of κ_t, which is here set equal to 0.9, leading to the same steady state share of eligible loans ($\kappa l/y = 1/3$) as in the benchmark model.[34] In addition to the set of parameter values from above, we apply the values $\epsilon_b = 1.7$ and $\epsilon_l = 0.3$, which are mainly chosen to facilitate the illustration of the effects via impulse responses, while the constraints on the policy instruments are again not binding for the cases under consideration.

The steady state allocations under both types of policy regimes (*opt*, *std*) differ from the first

[34] The instruments R_t^m and κ_t^B are again adjusted in a state-contingent way to implement the policy plan.

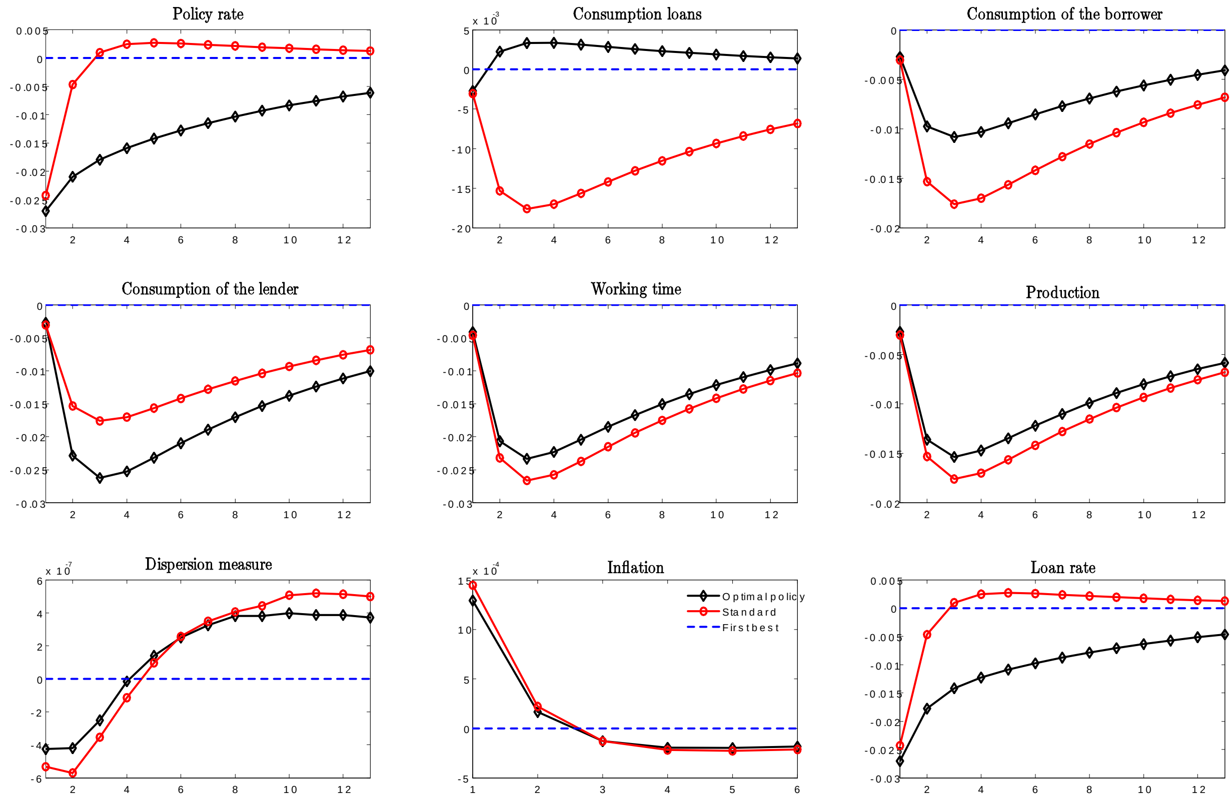

Figure 6: Responses (in % dev. from the steady states) to a cost push shock for the consumption loans model

best allocation. The steady state inflation rates are almost identical and close to one, whereas the steady state loan rate under optimal policy is smaller – and thus less distortive – than under non-rationed money supply; the difference of 1.4% for annualized rates being comparable to the benchmark model (see Figure 2). The steady state values for consumption of both types of agents and working time are all smaller than under the first best and differ from the latter by a few basis points. Under the non-rationing policy regime this difference is larger than under optimal policy for the consumption of the borrower, $\Delta c_b^{std}/\Delta c_b^{opt} = 2.4$ (where $\Delta c_b^x = c_b^x - c_b^*$ for $x \in \{opt, std\}$), as well as for working time and therefore output, $\Delta y^{std}/\Delta y^{opt} = 1.13$, while the opposite holds for the consumption of the lender, $\Delta c_l^{std}/\Delta c_l^{opt} = 0.75$. The latter is reflected by the share of loans under the optimal policy being much larger than under non-rationed money supply ($L_l^{opt}/L_l^{std} = 1.81$). The steady state allocations under both policies imply a gain from money rationing (measured as deviations from first best) of $Loss^{std}/Loss^{opt} = 1.17$, which is much smaller than in the benchmark model, where $Loss^{std}/Loss^{opt} = 1.8$ (see Figure 1 for $\gamma = 1$ and $\phi = 0.8$). This is mainly due to the fact that the policy instruments under money rationing are, here, less effective in reducing macroeconomic distortions than in the benchmark model, where firms' real marginal costs could be more directly stabilized via the firms' borrowing costs.

To further illustrate short-run effects from money rationing, we compute impulse responses to aggregate shocks. Figure 5 presents the impulse responses to a positive productivity shock for the optimal policy (solid black line with diamonds), the optimal policy under non-rationed money supply (red solid circled line), and the first best allocation (blue dashed line). The allocations under both policy regimes are hardly distinguishable, while they markedly differ from the first best allocation. A similar result applies for the inflation rate, whereas the response of the dispersion measure s_t is clearly less pronounced under the optimal policy regime than under the policy regime with non-rationed money supply. Like in the benchmark model, the policy rate under optimal policy is reduced to a smaller extent than under the non-rationing regime. The implied less pronounced reduction of the loan rate under the optimal policy is associated with an increase in money supplied against loans, which rise in both regimes (with a higher mean value under optimal policy).

Figure 6 further shows impulse responses to a cost-push shock. In contrast to the benchmark model, these shocks can not be neutralized under optimal policy, such that first best cannot be reproduced (as in Figure 4). Nevertheless, the optimal policy regime is overall more successful in stabilizing prices and the allocation than the non-rationing regime. Like in the benchmark model, the policy rate and the loan rate fall in response to the cost-push shock and the responses are more pronounced under optimal policy. Given that the decline in the policy rate is more persistent than the decline in the loan rate under optimal policy, the ratio R_t^L/R_t^m and therefore the lenders' willingness to supply cash increase after the initial period. In accordance with (F.5), this leads to an increase in loans and a difference in both types' consumption levels that is more pronounced than under non-rationed money supply.

G Further Details on the Consumption Loans Model

In this Appendix, we define the competitive equilibrium for the model version with consumption loans and we examine optimal monetary policy. It should be noted that the valuation of income $\lambda_t = \mu_t \chi n_{i,t}^{\eta}/w_t$ is identical for both types of households, which both exhibit a mass of 0.5. A competitive equilibrium can be defined as follows:

Definition 3 *A competitive equilibrium of the consumption loans model is a set of sequences* $\{c_{b,t}, c_{l,t}, n_{b,t}, n_{l,t}, n_t, l_{b,t}, l_{l,t}, i_{b,t}, i_{l,t}, m_t^L, m_{b,t}^H, m_{l,t}^H, m_t^H, b_{b,t}, b_{l,t}, b_t, w_t, mc_t, \tilde{Z}_t, s_t, \pi_t, R_t^L\}_{t=0}^{\infty}$ *satisfying*

$$n_{l,t} = n_{b,t} \quad \text{(G.1)}$$

$$\chi n_{b,t}^{\eta} = (w_t/\mu_t)\beta E_t[0.5(\epsilon_b c_{b,t+1}^{-\sigma} + \epsilon_l c_{l,t+1}^{-\sigma})/\pi_{t+1}], \quad \text{(G.2)}$$

$$1/R_t^L = \left(c_{b,t}^{\sigma}/\epsilon_{b,t}\right)\beta E_t[0.5(\epsilon_b c_{b,t+1}^{-\sigma} + \epsilon_l c_{l,t+1}^{-\sigma})/\pi_{t+1}], \quad \text{(G.3)}$$

$$1/R_t^L = (\kappa_t/R_t^m) + (1-\kappa_t)\beta\left(c_{l,t}^{\sigma}/\epsilon_{l,t}\right)E_t[0.5(\epsilon_b c_{b,t+1}^{-\sigma} + \epsilon_l c_{l,t+1}^{-\sigma})/\pi_{t+1}], \quad \text{(G.4)}$$

$$c_{b,t} = i_{b,t} + m_{b,t-1}^H \pi_t^{-1} - \left(l_{b,t}/R_t^L\right) \text{ if } \psi_{b,t} = (1-1/R_t^L)\epsilon_{b,t}c_{b,t}^{-\sigma} > 0 \quad \text{(G.5)}$$
$$\text{or } c_{b,t} > i_{b,t} + m_{b,t-1}^H \pi_t^{-1} - \left(l_{b,t}/R_t^L\right) \text{ if } \psi_{b,t} = 0,$$

$$c_{l,t} = i_{l,t} + m_{l,t-1}^H \pi_t^{-1} - \left(l_{l,t}/R_t^L\right) \text{ if } \psi_{l,t} = (R_t^m - 1)\left(\mu_t \chi n_{l,t}{}^{\eta}/a_t\right) + R_t^m \eta_{l,t} > 0 \quad \text{(G.6)}$$
$$\text{or } c_{l,t} > i_{l,t} + m_{l,t-1}^H \pi_t^{-1} - \left(l_{l,t}/R_t^L\right) \text{ if } \psi_{l,t} = 0,$$

$$R_t^m i_{l,t} = \kappa_t l_{l,t} + \kappa_t^B b_{l,t-1}\pi_t^{-1} \text{ if } \eta_{l,t} = (\epsilon_{l,t}c_{l,t}^{-\sigma}/R_t^m) - \beta E_t[0.5(\epsilon_b c_{b,t+1}^{-\sigma} + \epsilon_l c_{l,t+1}^{-\sigma})/\pi_{t+1}] > 0 \quad \text{(G.7)}$$
$$\text{or } R_t^m i_{l,t} < \kappa_t l_{l,t} + \kappa_t^B b_{l,t-1}\pi_t^{-1} \text{ if } \eta_{l,t} = 0,$$

$$R_t^m i_{b,t} = \kappa_t^B b_{b,t-1}\pi_t^{-1} \text{ if } \eta_{b,t} = (\epsilon_{b,t}c_{b,t}^{-\sigma}/R_t^m) - \beta E_t[0.5(\epsilon_b c_{b,t+1}^{-\sigma} + \epsilon_l c_{l,t+1}^{-\sigma})/\pi_{t+1}] > 0 \quad \text{(G.8)}$$
$$\text{or } R_t^m i_{b,t} < \kappa_t^B b_{b,t-1}\pi_t^{-1} \text{ if } \eta_{b,t} = 0,$$

$$R_t^m m_t^L = \kappa_t l_{l,t} \text{ if } \eta_{l,t} > 0 \text{ or } R_t^m m_t^L < \kappa_t l_{l,t} \text{ if } \eta_{l,t} = 0, \quad \text{(G.9)}$$

$$0 = (1-\tau^n)w_t - mc_t a_t \alpha n_t^{\alpha-1} \quad \text{(G.10)}$$

$$Z_{1,t}/Z_{2,t} = \tilde{Z}_t(\varepsilon-1)/\varepsilon, \text{ where } Z_{1,t} = (\mu_t \chi n_{b,t}^{\eta}/w_t)(a_t n_t^{\alpha}/s_t)mc_t + \phi\beta E_t \pi_{t+1}^{\varepsilon} Z_{1,t+1} \quad \text{(G.11)}$$
$$\text{and } Z_{2,t} = (\mu_t \chi n_{b,t}^{\eta}/w_t)(a_t n_t^{\alpha}/s_t) + \phi\beta E_t \pi_{t+1}^{\varepsilon-1} Z_{2,t+1}, \quad \text{(G.12)}$$

$$1 = (1-\phi)(\tilde{Z}_t)^{1-\varepsilon} + \phi\pi_t^{\varepsilon-1}, \quad \text{(G.13)}$$

$$s_t = (1-\phi)\tilde{Z}_t^{-\varepsilon} + \phi s_{t-1}\pi_t^{\varepsilon}, \quad \text{(G.14)}$$

$$a_t n_t^{\alpha}/s_t = c_{l,t} + c_{b,t}, \quad \text{(G.15)}$$

$$n_t = n_{l,t} + n_{b,t}, \quad \text{(G.16)}$$

$$-l_{b,t} = l_{l,t}, \quad \text{(G.17)}$$

$$m_{b,t}^H = m_{l,t}^H, \quad \text{(G.18)}$$

$$b_t = b_{b,t} + b_{l,t} \quad \text{(G.19)}$$

$$m_t^H = m_{b,t}^H + m_{l,t}^H \quad \text{(G.20)}$$

$$i_{b,t} = (1+\Omega_t)m_{b,t}^H - m_{b,t-1}^H, \quad \text{(G.21)}$$

$$i_{l,t} = (1+\Omega_t)m_{l,t}^H - m_{l,t-1}^H + m_{l,t}^L, \quad \text{(G.22)}$$

$$0 = \left(b_t + m_t^H\right) - \Gamma\left(b_{t-1} + m_{t-1}^H\right)/\pi_t, \quad \text{(G.23)}$$

the transversality conditions, a monetary policy setting $\{R_t^m \geq 1,\ \kappa_t^B \in (0,1],\ \kappa_t \in [0,1],\ \Omega_t \geq 0\}_{t=0}^{\infty}$*, a subsidy* $\tau^n = 1 - \frac{\varepsilon-1}{\varepsilon\mu}$*, given* $\{a_t, \mu_t\}_{t=0}^{\infty}$*,* $m_{b,-1}^H = m_{l,-1}^H > 0$*,* $b_{b,-1} = b_{l,-1} > 0$*,* $b_{-1} = b_{b,-1} + b_{l,-1} > 0$*,* $m_{-1}^H = m_{b,-1}^H + m_{l,-1}^H > 0$*, and* $s_{-1} = 1$*.*

For the analysis of optimal policy, we consider the first best allocation as a reference case. Using $n_t = n_{l,t} + n_{b,t}$ and $n_{l,t} = n_{b,t}$, the social planer problem can be summarized as

$$\max_{\{c_{l,t}, c_{b,t}, n_t, n_{j,t}, y_{k,t}\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t \left\{0.5\left[\epsilon_{b,t}(c_{b,t}^{1-\sigma}-1)+\epsilon_{l,t}(c_{l,t}^{1-\sigma}-1)\right](1-\sigma)^{-1} - \chi\,(0.5n_t)^{1+\eta}\,(1+\eta)^{-1}\right\}$$

$$\text{s.t. } a_t\int_0^1 n_{j,t}^{\alpha}dj = \int_0^1 y_{k,t}dk,\ \int_0^1 n_{j,t}dj = n_t, \text{ and } \int_0^1 y_{k,t}^{\frac{\varepsilon-1}{\varepsilon}}dk = (\int_0^1 c_{b,t}di + \int_0^1 c_{l,t}di)^{\frac{\varepsilon-1}{\varepsilon}}$$

Like in the proof of Proposition 1, the first order conditions imply that choices for individuals are identical. They can further be simplified to $\epsilon_{b,t}c_{b,t}^{-\sigma} = \epsilon_{l,t}c_{l,t}^{-\sigma}$, $\chi\,(0.5n_t)^{\eta} = a_t\alpha n_t^{\alpha-1}\epsilon_{b,t}c_{b,t}^{-\sigma}$, and $c_{l,t} + c_{b,t} = a_t n_t^{\alpha}$, which imply the following solution for the first best allocation

$$\begin{aligned} c_{b,t} &= a_t^{\frac{1+\eta}{1-\alpha+\eta+\alpha\sigma}}\left[\alpha\epsilon_{b,t}/(\chi 0.5^{\eta})\right]^{\frac{\alpha}{1-\alpha+\eta+\alpha\sigma}}\left[1+(\epsilon_{l,t}/\epsilon_{b,t})^{\frac{1}{\sigma}}\right]^{-\frac{1-\alpha+\eta}{1-\alpha+\eta+\alpha\sigma}}, \\ c_{l,t} &= (\epsilon_{l,t}/\epsilon_{b,t})^{\frac{1}{\sigma}}\,c_{b,t},\ \ n_t = (c_t/a_t)^{1/\alpha}. \end{aligned} \tag{G.24}$$

To identify the optimal monetary policy, we consider the competitive equilibrium as given in Definition 3 and eliminate w_t, $n_{l,t}$, $n_{b,t}$, $l_{b,t}$, $m_{l,t}^H$, $m_{b,t}^H$, m_t^L, $b_{b,t}$, $b_{l,t}$, $i_{b,t}$, and $i_{l,t}$, such that the set of equilibrium conditions is reduced to a system in mc_t, $\tilde{Z}_t$, $Z_{1,t}$, $Z_{2,t}$, s_t, $c_{b,t}$, $c_{l,t}$, n_t, m_{t-1}^H, π_t, $l_{l,t}$, R_t^L, and b_{t-1} satisfying (G.11), (G.13)-(G.15), (G.23),

$$\epsilon_{b,t}c_{b,t}^{-\sigma} = R_t^L\,(1-\tau^n)\,\mu_t\chi\,(0.5n_t)^{\eta}\,/\left(mc_t a_t\alpha n_t^{\alpha-1}\right), \tag{G.25}$$

$$\mu_t\,(1-\tau^n)\,\chi\,(0.5n_t)^{\eta} = \beta E_t[0.5(\epsilon_b c_{b,t+1}^{-\sigma} + \epsilon_l c_{l,t+1}^{-\sigma})/\pi_{t+1}]\left(mc_t a_t\alpha n_t^{\alpha-1}\right), \tag{G.26}$$

$$c_{b,t} \leq 0.5m_{t-1}^H\pi_t^{-1} + \left(l_{l,t}/R_t^L\right) + \kappa_t^B 0.5b_{t-1}\pi_t^{-1}/R_t^m, \tag{G.27}$$

$$c_{l,t} \leq (\kappa_t l_{l,t}/R_t^m) + 0.5m_{t-1}^H\pi_t^{-1} - \left(l_{l,t}/R_t^L\right) + \kappa_t^B 0.5b_{t-1}\pi_t^{-1}/R_t^m, \tag{G.28}$$

$$1/R_t^L = (\kappa_t/R_t^m) + (1-\kappa_t)\,\beta\left(c_{l,t}^{\sigma}/\epsilon_{l,t}\right)E_t[0.5(\epsilon_b c_{b,t+1}^{-\sigma} + \epsilon_l c_{l,t+1}^{-\sigma})/\pi_{t+1}], \tag{G.29}$$

$$Z_{1,t} = \mu_t\,(1-\tau^n)\,(\chi/\alpha)\,0.5^{\eta}n_t^{1+\eta}s_t^{-1} + \phi\beta E_t\pi_{t+1}^{\varepsilon}Z_{1,t+1}\ , \tag{G.30}$$

$$Z_{2,t} = \mu_t\,(1-\tau^n)\,(\chi/\alpha)\,0.5^{\eta}n_t^{1+\eta}\,(mc_t s_t)^{-1} + \phi\beta E_t\pi_{t+1}^{\varepsilon-1}Z_{2,t+1}, \tag{G.31}$$

$$\kappa_t^B b_{t-1}/R_t^m \leq \pi_t\,(1+\Omega_t)\,m_t^H - m_{t-1}^H, \tag{G.32}$$

and the transversality conditions, given $\{R_t^m \geq 1, \kappa_t^B \in (0,1], \kappa_t \in [0,1], \Omega_t \geq 0\}_{t=0}^{\infty}$, $\tau^n = 1 - \frac{\varepsilon-1}{\varepsilon\mu}$, $\{a_t, \mu_t\}_{t=0}^{\infty}$, $b_{-1} > 0$, $m_{-1}^H > 0$, and $s_{-1} = 1$. To identify the optimal monetary policy under commitment, we consider the following policy problem, where we consider the reduced set of equilibrium conditions, i.e. (G.11), (G.13)-(G.15), (G.23), (G.25)-(G.32), and $b_t^T = b_t + m_t^H$, as constraints:

$$\max_{\{c_{b,t},c_{l,t},n_t,m_t^H,b_t,b_t^T,l_{l,t},mc_t,\tilde{Z}_t,Z_{1,t},Z_{2,t},s_t,\pi_t,R_t^L,\kappa_t^B,\kappa_t,R_t^m\}_{t=0}^\infty} \min_{\{\theta_{1,t},...\theta_{14,t}\}_{t=0}^\infty} \tag{G.33}$$

$$E\sum_{t=0}^{\infty}\beta^t\left[0.5\epsilon_{b,t}(c_{b,t}^{1-\sigma}-1)(1-\sigma)^{-1}+0.5\epsilon_{l,t}(c_{l,t}^{1-\sigma}-1)(1-\sigma)^{-1}-\chi(0.5n_t)^{1+\eta}(1+\eta)^{-1}\right]$$
$$+\theta_{1,t}\left[\mu_t(1-\tau^n)\chi(0.5n_t)^\eta\epsilon_{b,t}^{-1}c_{b,t}^\sigma-mc_ta_t\alpha n_t^{\alpha-1}/R_t^L\right]$$
$$+\theta_{2,t}\left[\mu_t(1-\tau^n)\chi(0.5n_t)^\eta/\left(mc_ta_t\alpha n_t^{\alpha-1}\right)-\beta E_t[0.5(\epsilon_bc_{b,t+1}^{-\sigma}+\epsilon_lc_{l,t+1}^{-\sigma})/\pi_{t+1}]\right]$$
$$+\theta_{3,t}\left[(1/R_t^L)-(1-\kappa_t)\beta\left(c_{l,t}^\sigma/\epsilon_{l,t}\right)E_t[0.5(\epsilon_bc_{b,t+1}^{-\sigma}+\epsilon_lc_{l,t+1}^{-\sigma})/\pi_{t+1}]-\kappa_t/R_t^m\right]$$
$$+\theta_{4,t}\left[0.5(1+\Omega_t)m_t^H+\left(l_{l,t}/R_t^L\right)-c_{b,t}\right]+\theta_{5,t}\left[b_t-b_t^T+m_t^H\right]$$
$$+\theta_{6,t}\left[0.5(1+\Omega_t)m_t^H-\left(l_{l,t}/R_t^L\right)+(\kappa_tl_{l,t}/R_t^m)-c_{l,t}\right]+\theta_{7,t}\left[a_tn_t^\alpha/s_t-c_{l,t}-c_{b,t}\right]$$
$$+\theta_{8,t}\left[\tilde{Z}_t(\varepsilon-1)/\varepsilon-Z_{1,t}/Z_{2,t}\right]+\theta_{9,t}\left[(1-\phi)(\tilde{Z}_t)^{1-\varepsilon}+\phi\pi_t^{\varepsilon-1}-1\right]$$
$$+\theta_{10,t}\left[s_t-(1-\phi)\tilde{Z}_t^{-\varepsilon}-\phi s_{t-1}\pi_t^\varepsilon\right]$$
$$+\theta_{11,t}\left[Z_{1,t}-\mu_t(1-\tau^n)(\chi/\alpha)0.5^\eta n_t^{1+\eta}s_t^{-1}-\phi\beta E_t\pi_{t+1}^\varepsilon Z_{1,t+1}\right]$$
$$+\theta_{12,t}\left[Z_{2,t}-\mu_t(1-\tau^n)(\chi/\alpha)0.5^\eta n_t^{1+\eta}(mc_ts_t)^{-1}-\phi\beta E_t\pi_{t+1}^{\varepsilon-1}Z_{2,t+1}\right]$$
$$+\theta_{13,t}\left[b_t^T-\Gamma b_{t-1}^T/\pi_t\right]+\theta_{14,t}\left[\kappa_t^Bb_{t-1}/\left(R_t^m\pi_t\right)-(1+\Omega_t)m_t^H+m_{t-1}^H\pi_t^{-1}\right].$$

Like in the proof of Proposition 4, we show that several constraints in (G.33) are not binding: The first order condition for κ_t^B is given by $\theta_{14,t}b_{t-1}/\left(R_t^m\pi_t\right)=0\ \forall t\geq 0$ and implies $\theta_{14,t}=0$, while the first order condition for b_t, $\theta_{5,t}+\beta E_t\theta_{14,t+1}\kappa_{t+1}^B/R_{t+1}^m\pi_{t+1}=0\ \forall t\geq 0$, implies $\theta_{5,t}=0$. The first order condition for m_t^H, $\theta_{5,t}+\theta_{4,t}0.5(1+\Omega_t)+\theta_{6,t}0.5(1+\Omega_t)-(1+\Omega_t)\theta_{14,t}+\beta E_t\theta_{14,t+1}/\pi_{t+1}=0$ $\forall t\geq 0$, implies the sum $\theta_{4,t}+\theta_{6,t}$ to equal zero, $\theta_{4,t}+\theta_{6,t}=0$. The latter and the first order condition for consumption loans, $\theta_{4,t}/R_t^L+\theta_{6,t}\left(\kappa_t/R_t^m-1/R_t^L\right)=0\ \forall t\geq 0$, imply that both multipliers equal zero, $\theta_{4,t}=0$ and $\theta_{6,t}=0$, if $\kappa_t>0$. The constraint (A.12) is then irrelevant for the policy problem (and the first order condition for b_t^T, $\theta_{5,t}=\theta_{13,t}-\Gamma\beta E_t\theta_{13,t+1}/\pi_{t+1}$, is consistent with $\theta_{13,t}=0$). For

$$R_t^m<\frac{\epsilon_{l,t}c_{l,t}^{-\sigma}}{\beta E_t[(0.5\epsilon_bc_{b,t+1}^{-\sigma}+0.5\epsilon_lc_{l,t+1}^{-\sigma})/\pi_{t+1}]} \tag{G.34}$$

the first order condition for κ_t, which requires $\kappa_t>0$ and reads $\theta_{3,t}[\beta(c_{l,t}^\sigma/\epsilon_{l,t})E_t[0.5(\epsilon_bc_{b,t+1}^{-\sigma}+\epsilon_lc_{l,t+1}^{-\sigma})/\pi_{t+1}]-1/R_t^m]=\theta_{6,t}l_{l,t}/R_t^m\ \forall t\geq 0$, implies $\theta_{3,t}=0$. Suppose that $\kappa_t>0$ and (G.34) are satisfied. The first order condition for R_t^L, $\theta_{1,t}mc_ta_t\alpha n_t^{\alpha-1}\left(1/R_t^L\right)^2-\theta_{3,t}\left(1/R_t^L\right)^2+\theta_{4,t}l_{l,t}\left(1/R_t^L\right)^2-\theta_{6,t}l_{l,t}\left(1/R_t^L\right)^2=0\ \forall t\geq 0$, then implies $\theta_{1,t}=0$. In contrast to the benchmark model, further multipliers on the constraints cannot be verified to equal zero (see proof of Proposition 4). Thus, if $\kappa_t>0$ and (G.34) are satisfied, the collateral constraint (27) is binding and the policy problem can be summarized as

$$\max_{\{c_{b,t},c_{l,t},n_t,mc_t,\tilde{Z}_t,Z_{1,t},Z_{2,t},s_t,\pi_t,\}_{t=0}^{\infty}} \min_{\{\lambda_{1,t},...\lambda_{7,t}\}_{t=0}^{\infty}} \tag{G.35}$$

$$E\sum_{t=0}^{\infty}\beta^t\left[0.5\epsilon_{b,t}(c_{b,t}^{1-\sigma}-1)(1-\sigma)^{-1}+0.5\epsilon_{l,t}(c_{l,t}^{1-\sigma}-1)(1-\sigma)^{-1}-\chi(0.5n_t)^{1+\eta}(1+\eta)^{-1}\right]$$
$$+\lambda_{1,t}\left[\mu_t(1-\tau^n)\chi 0.5^\eta n_t^{\eta+1-\alpha}/(mc_t a_t\alpha)-\beta E_t[0.5(\epsilon_b c_{b,t+1}^{-\sigma}+\epsilon_l c_{l,t+1}^{-\sigma})/\pi_{t+1}]\right]$$
$$+\lambda_{2,t}\left[a_t n_t^\alpha/s_t-c_{l,t}-c_{b,t}\right]+\lambda_{3,t}[s_t-\phi s_{t-1}\pi_t^\varepsilon-(1-\phi)^{\frac{1}{1-\varepsilon}}\left(1-\phi\pi_t^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}}]$$
$$+\lambda_{4,t}\left[(1-\phi)(\tilde{Z}_t)^{1-\varepsilon}+\phi\pi_t^{\varepsilon-1}-1\right]+\lambda_{5,t}\left[\tilde{Z}_t(\varepsilon-1)/\varepsilon-Z_{1,t}/Z_{2,t}\right]$$
$$+\lambda_{6,t}\left[Z_{1,t}-\mu_t(1-\tau^n)(\chi/\alpha)0.5^\eta n_t^{1+\eta}s_t^{-1}-\phi\beta E_t\pi_{t+1}^\varepsilon Z_{1,t+1}\right]$$
$$+\lambda_{7,t}\left[Z_{2,t}-\mu_t(1-\tau^n)(\chi/\alpha)0.5^\eta n_t^{1+\eta}(mc_t s_t)^{-1}-\phi\beta E_t\pi_{t+1}^{\varepsilon-1}Z_{2,t+1}\right].$$

Neglecting the conditions for $t=0$, the solution to the policy problem (G.35) has to satisfy the following first order conditions:

$$0=0.5\epsilon_{b,t}c_{b,t}^{-\sigma}+\lambda_{1,t-1}\epsilon_{b,t}0.5\sigma\left(c_{b,t}^{-\sigma-1}/\pi_t\right)-\lambda_{2,t},$$
$$0=0.5\epsilon_{l,t}c_{l,t}^{-\sigma}+\lambda_{1,t-1}\epsilon_{l,t}0.5\sigma\left(c_{l,t}^{-\sigma-1}/\pi_t\right)-\lambda_{2,t},$$
$$0=-\chi 0.5^{1+\eta}n_t^{\eta}+[\lambda_{1,t}(\eta+1-\alpha)\mu_t(1-\tau^n)\chi 0.5^\eta n_t^{\eta-\alpha}/(mc_t\alpha a_t)]+(\lambda_{2,t}a_t\alpha n_t^{\alpha-1}/s_t)$$
$$-\lambda_{6,t}(1+\eta)\mu_t(1-\tau^n)(\chi/\alpha)0.5^\eta n_t^\eta s_t^{-1}-\lambda_{7,t}(1+\eta)\mu_t(1-\tau^n)(\chi/\alpha)0.5^\eta n_t^\eta(mc_t s_t)^{-1},$$

$$0=[\lambda_{1,t-1}\left(0.5\epsilon_b c_{b,t}^{-\sigma}+0.5\epsilon_l c_{l,t}^{-\sigma}\right)/\pi_t^2]+\lambda_{4,t}(\varepsilon-1)\phi\pi_t^{\varepsilon-2}$$
$$-\lambda_{6,t-1}\phi\varepsilon\pi_t^{\varepsilon-1}Z_{1,t}-\lambda_{7,t-1}\phi(\varepsilon-1)\pi_t^{\varepsilon-2}Z_{2,t}$$
$$+\lambda_{3,t}\left[-\phi s_{t-1}\varepsilon\pi_t^{\varepsilon-1}-(1-\phi)^{\frac{1}{1-\varepsilon}}\frac{\varepsilon}{\varepsilon-1}\left(1-\phi\pi_t^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}\left(-(\varepsilon-1)\phi\pi_t^{\varepsilon-2}\right)\right],$$
$$0=-(\lambda_{2,t}a_t n_t^\alpha/s_t^2)+\lambda_{3,t}-\beta E_t\lambda_{3,t+1}\phi\pi_{t+1}^\varepsilon$$
$$+\lambda_{6,t}\mu_t(1-\tau^n)(\chi/\alpha)0.5^\eta n_t^{1+\eta}s_t^{-2}+\lambda_{7,t}\mu_t(1-\tau^n)(\chi/\alpha)0.5^\eta n_t^{1+\eta}mc_t^{-1}s_t^{-2}$$
$$-[\lambda_{1,t}\mu_t(1-\tau^n)\chi 0.5^\eta n_t^{\eta+1-\alpha}/\left(mc_t^2\alpha a_t\right)]+\lambda_{7,t}\mu_t(1-\tau^n)(\chi/\alpha)0.5^\eta n_t^{1+\eta}mc_t^{-2}s_t^{-1},$$
$$0=-(\lambda_{5,t}/Z_{2,t})+\lambda_{6,t}-\lambda_{6,t-1}\phi\pi_t^\varepsilon,$$
$$0=\lambda_{5,t}(Z_{1,t}/Z_{2,t}^2)+\lambda_{7,t}-\lambda_{7,t-1}\phi\pi_t^{\varepsilon-1},$$
$$0=\lambda_{4,t}(1-\phi)(1-\varepsilon)(\tilde{Z}_t)^{-\varepsilon}+\lambda_{5,t}(\varepsilon-1)/\varepsilon,$$

as well as $0=[\mu_t(1-\tau^n)\chi 0.5^\eta n_t^{\eta+1-\alpha}/(mc_t a_t\alpha)]-\beta E_t[0.5(\epsilon_b c_{b,t+1}^{-\sigma}+\epsilon_l c_{l,t+1}^{-\sigma})/\pi_{t+1}]$, (G.11), (G.13)-(G.15), (G.30), and (G.31). The steady state of the solution to (G.35), where all exogenous and endogenous variables satisfy $x_t=x_{t-1}=x_{t+1}=x$, can be reduced to a set

$\{c_b, c_l, n, \pi, s, \lambda_1, \lambda_3\}$ satisfying

$$
\begin{aligned}
0 =& \epsilon_l c_l^{-\sigma}\left(1+\sigma c_l^{-1}\lambda_1/\pi\right) - \epsilon_b c_b^{-\sigma}\left(1+\sigma c_b^{-1}\lambda_1/\pi\right), \\
0 =& \frac{\pi}{\beta}0.5\left(1+\sigma c_b^{-1}\lambda_1/\pi\right)\left(\alpha n^{\alpha}/s\right) + \lambda_1 \frac{0.5\epsilon_b c_b^{-\sigma}+0.5\epsilon_l c_l^{-\sigma}}{\epsilon_b c_b^{-\sigma}}\left(\eta+1-\alpha+(1+\eta)\Phi(\pi)\right) \\
& -\frac{\chi n^{1+\eta}0.5^{1+\eta}}{\epsilon_b c_b^{-\sigma}}\frac{\pi}{\beta}, \\
0 =& \lambda_1 \frac{0.5\epsilon_b c_b^{-\sigma}+0.5\epsilon_l c_l^{-\sigma}}{\epsilon_b c_b^{-\sigma}}\Phi(\pi) + \frac{\pi}{\beta}0.5\left(1+\sigma c_b^{-1}\lambda_1/\pi\right)\left(n^{\alpha}/s\right) - \frac{\pi}{\beta}s\lambda_3\frac{c_b^{\sigma}}{\epsilon_b}\left(1-\beta\phi\pi^{\varepsilon}\right), \\
0 =& -\lambda_1 + \lambda_3\phi\varepsilon\pi^{\varepsilon}\frac{s}{0.5\epsilon_b c_b^{-\sigma}+0.5\epsilon_l c_l^{-\sigma}}\frac{\pi-1}{1-\phi\pi^{\varepsilon-1}} + \lambda_1\beta\frac{\varepsilon\phi\pi^{\varepsilon-1}}{1-\phi\beta\pi^{\varepsilon-1}}\frac{1-\pi}{1-\phi\pi^{\varepsilon}}, \\
0 =& \left(1-\tau^n\right)\mu\frac{\varepsilon}{\varepsilon-1}\frac{\pi}{\beta}\left(\frac{1-\phi\pi^{\varepsilon-1}}{1-\phi}\right)^{\frac{1}{\varepsilon-1}}\frac{\left(1-\phi\beta\pi^{\varepsilon-1}\right)}{\left(1-\phi\beta\pi^{\varepsilon}\right)} - \frac{0.5\epsilon_b c_b^{-\sigma}+0.5\epsilon_l c_l^{-\sigma}}{(\chi/\alpha)\,0.5^{\eta}n^{\eta+1-\alpha}}, \\
0 =& (1-\phi)^{\frac{1}{1-\varepsilon}}\frac{\left(1-\phi\pi^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}}}{\left(1-\phi\pi^{\varepsilon}\right)} - s, \\
0 =& c_l + c_b - n^{\alpha}/s,
\end{aligned}
$$

where $\Phi(\pi) = \frac{1-\phi\beta\pi^{\varepsilon}}{1-\phi\beta\pi^{\varepsilon-1}}\frac{1-\phi\pi^{\varepsilon-1}}{1-\phi\pi^{\varepsilon}} - 1$.

When (G.34) is not satisfied, money supply is not effectively rationed. Then, (G.25) and (G.26), which can be combined to $1/R_t^L = \frac{1}{\epsilon_{b,t}c_{b,t}^{-\sigma}}\beta E_t\frac{0.5\epsilon_b c_{b,t+1}^{-\sigma}+0.5\epsilon_l c_{l,t+1}^{-\sigma}}{\pi_{t+1}}$, and (G.29) imply $\epsilon_{l,t}c_{l,t}^{-\sigma} = \epsilon_{b,t}c_{b,t}^{-\sigma}$. The policy problem can then be summarized as

$$
\begin{aligned}
& \max_{\{c_{b,t},c_{l,t},n_t,mc_t,\tilde{Z}_t,Z_{1,t},Z_{2,t},s_t,\pi_t,\}_{t=0}^{\infty}} \quad \min_{\{\lambda_{0,t},\dots\lambda_{7,t}\}_{t=0}^{\infty}} \qquad \text{(G.36)} \\
& E\sum_{t=0}^{\infty}\beta^t\left[0.5\epsilon_{b,t}(c_{b,t}^{1-\sigma}-1)(1-\sigma)^{-1} + 0.5\epsilon_{l,t}(c_{l,t}^{1-\sigma}-1)(1-\sigma)^{-1} - \chi\left(0.5n_t\right)^{1+\eta}(1+\eta)^{-1}\right] \\
& +\lambda_{0,t}\left[0.5\epsilon_{l,t}c_{l,t}^{-\sigma} - 0.5\epsilon_{b,t}c_{b,t}^{-\sigma}\right] \\
& +\lambda_{1,t}\left[\mu_t\left(1-\tau^n\right)\chi 0.5^{\eta}n_t^{\eta+1-\alpha}/\left(mc_t a_t\alpha\right) - \beta E_t[0.5(\epsilon_b c_{b,t+1}^{-\sigma}+\epsilon_l c_{l,t+1}^{-\sigma})/\pi_{t+1}]\right] \\
& +\lambda_{2,t}\left[a_t n_t^{\alpha}/s_t - c_{l,t} - c_{b,t}\right] + \lambda_{3,t}\left[s_t - \phi s_{t-1}\pi_t^{\varepsilon} - (1-\phi)^{\frac{1}{1-\varepsilon}}\left(1-\phi\pi_t^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right] \\
& +\lambda_{4,t}\left[(1-\phi)(\tilde{Z}_t)^{1-\varepsilon} + \phi\pi_t^{\varepsilon-1} - 1\right] + \lambda_{5,t}\left[\tilde{Z}_t\left(\varepsilon-1\right)/\varepsilon - Z_{1,t}/Z_{2,t}\right] \\
& +\lambda_{6,t}\left[Z_{1,t} - \mu_t\left(1-\tau^n\right)\left(\chi/\alpha\right)0.5^{\eta}n_t^{1+\eta}s_t^{-1} - \phi\beta E_t\pi_{t+1}^{\varepsilon}Z_{1,t+1}\right] \\
& +\lambda_{7,t}\left[Z_{2,t} - \mu_t\left(1-\tau^n\right)\left(\chi/\alpha\right)0.5^{\eta}n_t^{1+\eta}\left(mc_t s_t\right)^{-1} - \phi\beta E_t\pi_{t+1}^{\varepsilon-1}Z_{2,t+1}\right].
\end{aligned}
$$

Neglecting the conditions for $t = 0$, the solution to the policy problem (G.36) has to satisfy the following first order conditions

$$\begin{aligned}
0 &= 0.5\epsilon_{b,t}c_{b,t}^{-\sigma} - 0.5\lambda_{0,t}\epsilon_{b,t}\sigma c_{b,t}^{-\sigma-1} + \lambda_{1,t-1}\epsilon_{b,t}0.5\sigma\left(c_{b,t}^{-\sigma-1}/\pi_t\right) - \lambda_{2,t},\\
0 &= 0.5\epsilon_{l,t}c_{l,t}^{-\sigma} + 0.5\lambda_{0,t}\sigma\epsilon_{l,t}c_{l,t}^{-\sigma-1} + \lambda_{1,t-1}\epsilon_{l,t}0.5\sigma\left(c_{l,t}^{-\sigma-1}/\pi_t\right) - \lambda_{2,t},\\
0 &= -\chi 0.5^{1+\eta}n_t^{\eta} + [\lambda_{1,t}(\eta+1-\alpha)\mu_t(1-\tau^n)\chi 0.5^{\eta}n_t^{\eta-\alpha}/(mc_t\alpha a_t)] + (\lambda_{2,t}a_t\alpha n_t^{\alpha-1}/s_t)\\
&\quad -\lambda_{6,t}(1+\eta)\mu_t(1-\tau^n)(\chi/\alpha)0.5^{\eta}n_t^{\eta}s_t^{-1} - \lambda_{7,t}(1+\eta)\mu_t(1-\tau^n)(\chi/\alpha)0.5^{\eta}n_t^{\eta}(mc_ts_t)^{-1},\\
0 &= [\lambda_{1,t-1}\left(0.5\epsilon_b c_{b,t}^{-\sigma} + 0.5\epsilon_l c_{l,t}^{-\sigma}\right)/\pi_t^2] + \lambda_{4,t}(\varepsilon-1)\phi\pi_t^{\varepsilon-2}\\
&\quad -\lambda_{6,t-1}\phi\varepsilon\pi_t^{\varepsilon-1}Z_{1,t} - \lambda_{7,t-1}\phi(\varepsilon-1)\pi_t^{\varepsilon-2}Z_{2,t}\\
&\quad +\lambda_{3,t}\left[-\phi s_{t-1}\varepsilon\pi_t^{\varepsilon-1} - (1-\phi)^{\frac{1}{1-\varepsilon}}\frac{\varepsilon}{\varepsilon-1}\left(1-\phi\pi_t^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}\left(-(\varepsilon-1)\phi\pi_t^{\varepsilon-2}\right)\right],\\
0 &= -(\lambda_{2,t}a_tn_t^{\alpha}/s_t^2) + \lambda_{3,t} - \beta E_t\lambda_{3,t+1}\phi\pi_{t+1}^{\varepsilon}\\
&\quad +\lambda_{6,t}\mu_t(1-\tau^n)(\chi/\alpha)0.5^{\eta}n_t^{1+\eta}s_t^{-2} + \lambda_{7,t}\mu_t(1-\tau^n)(\chi/\alpha)0.5^{\eta}n_t^{1+\eta}mc_t^{-1}s_t^{-2},\\
0 &= -[\lambda_{1,t}\mu_t(1-\tau^n)\chi 0.5^{\eta}n_t^{\eta+1-\alpha}/\left(mc_t^2\alpha a_t\right)] + \lambda_{7,t}\mu_t(1-\tau^n)(\chi/\alpha)0.5^{\eta}n_t^{1+\eta}mc_t^{-2}s_t^{-1},\\
0 &= -(\lambda_{5,t}/Z_{2,t}) + \lambda_{6,t} - \lambda_{6,t-1}\phi\pi_t^{\varepsilon},\\
0 &= (\lambda_{5,t}Z_{1,t}/Z_{2,t}^2) + \lambda_{7,t} - \lambda_{7,t-1}\phi\pi_t^{\varepsilon-1},\\
0 &= \lambda_{4,t}(1-\phi)(1-\varepsilon)(\tilde{Z}_t)^{-\varepsilon} + \lambda_{5,t}(\varepsilon-1)/\varepsilon,
\end{aligned}$$

as well as $0 = \mu_t(1-\tau^n)\chi 0.5^{\eta}n_t^{\eta+1-\alpha}/(mc_ta_t\alpha) - \beta E_t\frac{\epsilon_b c_{b,t+1}^{-\sigma}}{\pi_{t+1}}$, $0 = \epsilon_{l,t}c_{l,t}^{-\sigma} - \epsilon_{b,t}c_{b,t}^{-\sigma}$, (G.11), (G.13)-(G.15), (G.30), and (G.31). The steady state of the solution to (G.36), can be reduced to a set $\{c_b, c_l, n, \pi, s, \lambda_0, \lambda_1, \lambda_3\}$ satisfying

$$\begin{aligned}
0 &= c_l^{-1}(\lambda_0 + \lambda_1/\pi) + c_b^{-1}(\lambda_0 - \lambda_1/\pi),\\
0 &= \frac{\pi}{\beta}0.5\left(1-\sigma c_b^{-1}(\lambda_0 - \lambda_1/\pi)\right)(\alpha n^{\alpha}/s) + \lambda_1(\eta+1-\alpha+(1+\eta)\Phi(\pi)) - \frac{\chi n^{1+\eta}0.5^{1+\eta}}{\epsilon_b c_b^{-\sigma}}\frac{\pi}{\beta},\\
0 &= \lambda_1\Phi(\pi) + \frac{\pi}{\beta}0.5\left(1-\sigma c_b^{-1}(\lambda_0 - \lambda_1/\pi)\right)(n^{\alpha}/s) - \frac{\pi}{\beta}s\lambda_3\frac{c_b^{\sigma}}{\epsilon_b}(1-\beta\phi\pi^{\varepsilon}),\\
0 &= -\lambda_1 + \lambda_3\phi\varepsilon\pi^{\varepsilon}\frac{s}{\epsilon_b c_b^{-\sigma}}\frac{\pi-1}{1-\phi\pi^{\varepsilon-1}} + \lambda_1\beta\frac{\varepsilon\phi\pi^{\varepsilon-1}}{1-\phi\beta\pi^{\varepsilon-1}}\frac{1-\pi}{1-\phi\pi^{\varepsilon}},\\
0 &= \mu(1-\tau^n)\frac{\varepsilon}{\varepsilon-1}\frac{\pi}{\beta}\left(\frac{1-\phi\pi^{\varepsilon-1}}{1-\phi}\right)^{\frac{1}{\varepsilon-1}}\frac{\left(1-\phi\beta\pi^{\varepsilon-1}\right)}{(1-\phi\beta\pi^{\varepsilon})} - \frac{\epsilon_b c_b^{-\sigma}}{(\chi/\alpha)0.5^{\eta}n^{\eta+1-\alpha}},\\
0 &= (1-\phi)^{\frac{1}{1-\varepsilon}}\frac{\left(1-\phi\pi^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}}}{(1-\phi\pi^{\varepsilon})} - s,\\
0 &= c_l + c_b - n^{\alpha}/s,\\
0 &= \epsilon_l c_l^{-\sigma} - \epsilon_b c_b^{-\sigma},
\end{aligned}$$

where $\Phi(\pi) = \frac{1-\phi\beta\pi^{\varepsilon}}{1-\phi\beta\pi^{\varepsilon-1}}\frac{1-\phi\pi^{\varepsilon-1}}{1-\phi\pi^{\varepsilon}} - 1$.